Statistics Denmark

Index theory

25th of January 2015

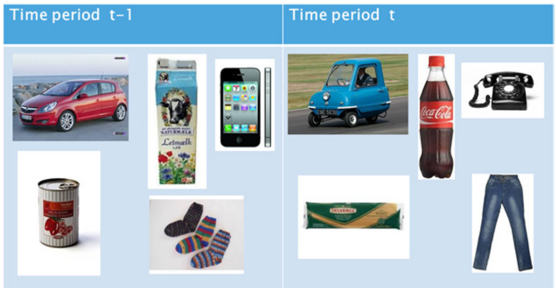
Index Theory

- Content:
  - Types of indices
  - Choice of the ideal index
  - Choice of elementary index
  - Higher level indices
  - Missing observations and quality adjustments

Types of indices

- Index:
  - Gives a simple expression for developments in values or highlights differences between groups

Types of indices – volume indices



Types of indices – volume indices

Index:
- A starting point is values:

V = P*Q

Types of indices – volume indices

Index of values:

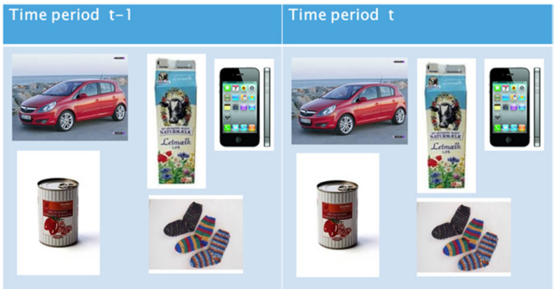
$$V_{0:t} = \frac{\sum_{i=1}^{N} p_t^i \cdot q_t^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i}$$

Types of indices – volume indices

- Example:

	p0	q0	p1	q1	p2	q2	p3	q3
Product A	10	6	11	6	12	5	13	5
Product B	7	5	7	6	6	7	6	6
Product C	8	10	9	9	9	9	10	8
Volume index	100,0		108,0		104,6		103,4	

Types of indices – price indices



Types of indices – price indices

Price indices:

Laspeyres:

$$P_{0:t}^{LA} = \frac{\sum_{i=1}^{N} p_t^i \cdot q_0^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i} = \sum_{i=1}^{N} \left(p_t^i \cdot \frac{q_0^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i} \right) = \sum_{i=1}^{N} \left(\frac{p_t^i}{p_0^i} \cdot \frac{p_0^i \cdot q_0^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i} \right)$$

$$= \sum_{i=1}^{N} w_0^i \cdot \left(\frac{p_t^i}{p_0^i} \right)$$

where $w_0^i = \frac{p_0^i \cdot q_0^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i}$ and $\sum_{i=1}^{N} w_0^i = 1$

Types of indices – price indices

Price indices:

Paasche:

$$P_{0:t}^{PA} = \frac{\sum_{i=1}^{N} p_t^i \cdot q_t^i}{\sum_{i=1}^{N} p_0^i \cdot q_t^i} = \left(\sum_{i=1}^{N} p_0^i \cdot q_t^i \frac{1}{\sum_{i=1}^{N} p_t^i \cdot q_t^i} \right)^{-1} = \left(\sum_{i=1}^{N} \frac{p_0^i}{p_t^i} \cdot \frac{p_t^i \cdot q_t^i}{\sum_{i=1}^{N} p_t^i \cdot q_t^i} \right)^{-1}$$

$$= \frac{1}{\sum_{i=1}^{N} \frac{p_0^i}{p_t^i} \cdot w_t^i}$$

where $w_t^i = \frac{p_t^i \cdot q_t^i}{\sum_{i=1}^{N} p_t^i \cdot q_t^i}$ and $\sum_{i=1}^{N} w_t^i = 1$

Types of indices – price indices

Price indices:

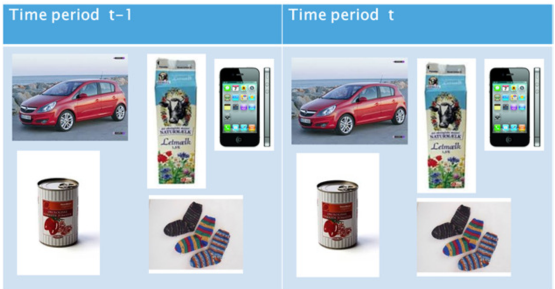
Fisher:

$$P_{0:t}^{FI} = \sqrt{P_{0:t}^{PA} \cdot P_{0:t}^{LA}}$$

Types of indices – price indices

	p0	q0	p1	q1	p2	q2	p3	q3
Product A	10	6	11	6	12	5	13	5
Product B	7	5	7	6	6	7	6	6
Product C	8	10	9	9	9	9	10	8
Laspeyres	100		109,1		109,7		118,9	
Paasche	100		108,6		107,0		116,0	
Fisher	100		108,9		108,4		117,4	

Types of indices – quantity indices



Types of indices

Quantity indices:
Laspeyres:

$$Q_{0:t}^{LA} = \frac{\sum_{i=1}^{N} p_0^i \cdot q_t^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i} = \sum_{i=1}^{N} \frac{q_t^i}{q_0^i} \frac{p_0^i \cdot q_0^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i} = \sum_{i=1}^{N} \frac{q_t^i}{q_0^i} w_0^i$$

Types of indices

Quantity indices:
Paasche:

$$Q_{0:t}^{PA} = \frac{\sum_{i=1}^{N} p_t^i \cdot q_t^i}{\sum_{i=1}^{N} p_t^i \cdot q_0^i} = \left(\sum_{i=1}^{N} \frac{q_0^i}{q_t^i} \frac{p_t^i \cdot q_t^i}{\sum_{i=1}^{N} p_t^i \cdot q_t^i} \right)^{-1} = \left(\sum_{i=1}^{N} \frac{q_0^i}{q_t^i} w_t^i \right)^{-1} = \frac{1}{\sum_{i=1}^{N} \frac{q_0^i}{q_t^i} w_t^i}$$

Types of indices

Quantity indices:
Fisher:

$$Q_{0:t}^{FI} = \sqrt{Q_{0:t}^{PA} \cdot Q_{0:t}^{LA}}$$

Types of indices – quantity indices

	p0	q0	p1	q1	p2	q2	p3	q3
Product A	10	6	11	6	12	5	13	5
Product B	7	5	7	6	6	7	6	6
Product C	8	10	9	9	9	9	10	8
Laspeyres	100		99,4		97,7		89,1	
Paasche	100		99,0		95,3		87,0	
Fisher	100		99,2		96,5		88,1	

Types of indices – implicit indices

Implicit indices:

$$P_{0:t}^{LA} \cdot Q_{0:t}^{PA} = \frac{\sum_{i=1}^{N} p_t^i \cdot q_0^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i} \cdot \frac{\sum_{i=1}^{N} p_t^i \cdot q_t^i}{\sum_{i=1}^{N} p_t^i \cdot q_0^i} = \frac{\sum_{i=1}^{N} p_t^i \cdot q_t^i}{\sum_{i=1}^{N} p_0^i \cdot q_0^i} = V_{0:t}$$

=> V / P = Q

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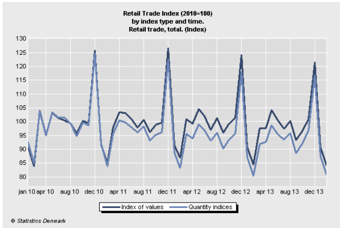
Types of indices

Implicit indices, example:

	Period 1		Period 2	
	Price	Quantity	Price	Quantity
Good A	12	10	11	16
Good B	5	15	5	13
Good C	10	10	10	8
Value at current prices	295		321	
Value at period 2 prices	285		321	
Value at period 1 quantities	295		285	
Value Index	100		108,8	
Paasche Quantity Index	100		112,6	
Laspeyres Price Index	100		96,6	
Value/Lasp. Price	100		112,6	

Types of indices

Implicit indices, Retail Trade Index:



Types of indices

Implicit indices, limitations:

The price index and the value index do often not use the same time periods => hence the implicit quantity index is not an exact quantity index

Types of indices – unit value indices

Unit value indices:

Unit price: $$E_t = \frac{\sum_{i=1}^{N} p_t^i \cdot q_t^i}{\sum_{i=1}^{N} q_t^i}$$

The index: $$P_t = \frac{E_t}{E_0}$$

Choice of index

	Purpose	
	Ideal index	
	Estimate	

Choice of index

Price index for fixed basket:

Lowe:

$$I_{0:t}^{Lo} = \frac{\sum p_t^j q_b^j}{\sum p_0^j q_b^j}$$

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Choice of index

Walsh:

$$I^{W}_{0:t} = \frac{\sum p^j_t \sqrt{q^j_0 \cdot q^j_t}}{\sum p^j_0 \sqrt{q^j_0 \cdot q^j_t}} = \sum w^j_W \cdot \left(\frac{p^j_t}{p^j_0}\right)$$

$$w^j_W = \frac{\sqrt{(w^j_0 \cdot w^j_t)/(p^j_t/p^j_0)}}{\sum \sqrt{(w^j_0 \cdot w^j_t)/(p^j_t/p^j_0)}}$$

Edgeworth:

$$I^{E}_{0:t} = \frac{\sum p^j_t \cdot (q^j_0 + q^j_t)/2}{\sum p^j_0 \cdot (q^j_0 + q^j_t)/2} = \sum w^j_E \cdot \left(\frac{p^j_t}{p^j_0}\right)$$

$$w^j_E = \frac{v^j_0 + \left(v^j_t / (p^j_t / p^j_0)\right)}{\sum \left(v^j_0 + \left(v^j_t / (p^j_t / p^j_0)\right)\right)}, v^j_t = \frac{p^j_t q^j_t}{\sum p^j_t q^j_t}$$

Choice of index

	p0	q0	p1	q1	p2	q2	p3	q3
Product A	10	6	11	6	12	5	13	5
Product B	7	5	7	6	6	7	6	6
Product C	8	10	9	9	9	9	10	8
Walsh	100		108,9		108,4		117,5	

Choice of index

Criteria for ideal indices:

1. Good statistical properties (superlative indices)
2. Easy to analyze and use

Ideal Index for the Danish Producer Price Index: Walsh index

Choice of index

Estimate of ideal index:

Most often in reality:

Lowe:

$$I^{Lo}_{0:t} = \frac{\sum p^j_t q^j_b}{\sum p^j_0 q^j_b} = \sum w^j_{b(0)} \left(\frac{p^j_t}{p^j_0}\right)$$

$$w^j_{b(0)} = \frac{w^j_b (p^j_0 / p^j_b)}{\sum w^j_b (p^j_0 / p^j_b)}$$

Young:

$$I^{Yo}_{0:t} = \sum w^j_b \left(\frac{p^j_t}{p^j_0}\right)$$

Choice of index

Estimate of ideal index:

Whether a Young or Lowe index is the best estimate for a Walsh index depends on whether w^j_b or $w^j_{b(0)}$ are the best estimates of the average budget shares in the Walsh index.

The Young Index is used in the Danish Producer Price Index.

Choice of elementary index

Most often in reality price indices are being calculated in a two-step procedure:

1. Elementary indices
2. Aggregate indices

Choice of elementary index

Products are grouped into elementary aggregates based on the following considerations:

- Elementary aggregates should have be significant economically or be of particular interest.
- They must have a clear and meaningful content. This is ensured by grouping products that are as homogeneous as possible.
- Products are collected in an elementary aggregate should show similar price trends, so that the expected variance is reduced.

Choice of elementary index

Carli: $$I^C_{0:t} = \frac{1}{n}\sum\left(\frac{p^i_t}{p^i_0}\right)$$

Dutot: $$I^D_{0:t} = \frac{\frac{1}{n}\sum p^i_t}{\frac{1}{n}\sum p^i_0} = \frac{\sum (p^i_t/p^i_0)\cdot p^i_0}{\sum p^i_0}$$

Jevons: $$I^J_{0:t} = \prod\left(\frac{p^i_t}{p^i_0}\right)^{1/n} = \frac{\prod (p^i_t)^{1/n}}{\prod (p^i_0)^{1/n}}$$

Choice of elementary index

Axiomatic Approach:

Proportionality test: I(p0,cp1) = cI(p0,p1)
Time reversal test: I(p0,p1) = 1/I(p1,p0)
Transitivity: I(p0,p2) = I(p0,p1) · I(p1,p2)
Changes in units of measurement tests:
I(cp0,cp1) = I(p0,p1), c = c1 ,.., cn > 0

Choice of elementary index

	January	February	March	April	May	June	July
	Prices						
Good A	6	6	7	6	6	6	6,6
Good B	7	7	6	7	7	7,2	7,7
Good C	2	3	4	5	2	3	2,2
Good D	5	5	5	4	5	5	5,5
Arithmethic mean	5	5,25	5,5	5,5	5	5,3	5,5
Geometric mean	4,53	5,01	5,38	5,38	4,53	5,05	4,98
Carli index – aritmetic mean of price relatives							
Monthly index	100	112,5	108,9	101,9	91,3	113,2	100,1
Chained monthly index	100	112,5	122,5	124,8	113,9	128,9	129
Direct index on January	100	112,5	125,6	132,5	100	113,2	110
Dutot index – ratio of unweighted arithmtic mean prices							
Monthly index	100	105	104,8	100	90,9	106	103,8
Chained monthly index	100	105	110	110	100	106	110
Direct index on January	100	105	110	110	100	106	110
Dutot index – ratio of unweighted geometric mean prices							
Monthly index	100	110,7	107,5	100	84,1	111,5	98,7
Chained monthly index	100	110,7	118,9	118,9	100	111,5	110
Direct index on January	100	110,7	118,9	118,9	100	111,5	110

Choice of elementary index

Jevons < Carli

Jevons <> Dutot

The Danish Producer Price Index is using the Jevons Index

Higher level indices

Lowe: $$I^{Lo}_{0:t} = \frac{\sum p^j_t q^j_b}{\sum p^j_0 q^j_b} = \sum w^j_{b(0)}\left(\frac{p^j_t}{p^j_0}\right)$$

$$w^j_{b(0)} = \frac{w^j_b\,(p^j_0/p^j_b)}{\sum w^j_b\,(p^j_0/p^j_b)}$$

Young: $$I^{Yo}_{0:t} = \sum w^j_b\left(\frac{p^j_t}{p^j_0}\right)$$

Higher level indices: $$I_{0:t} = \sum w^j_b \cdot I^j_{0:t}\ , \qquad \sum w^j_b = 1$$

Higher level indices

Chaining of indices:

$$\begin{aligned} I_{0:t} &= \sum w_b^j \cdot I_{0:k}^j \cdot \sum w_c^j \cdot I_{k:t}^j \\ &= I_{0:k} \cdot \sum w_c^j \cdot I_{k:t}^j \\ &= I_{0:k} \cdot I_{k:t} \end{aligned}$$

Higher level indices

Example, chaining of indices:

4th quarter=100	Q4	Weights	Q1	Q2	Q3	Q4	Weights	Q1	Q2	Q3	Q4
Elementary index A	100	0,4	102	104	102	104	0,45	104	105	105	106
Elementary index B	100	0,3	102	103	104	101	0,2	102	103	105	103
Elementary index C	100	0,3	100	99	99	97	0,35	96	96	95	97
Total index	100	1	101,4	102,2	101,7	101,0	1	100,8	101,5	101,5	102,3

Higher level indices

Decomposition:

$$\frac{w_b^j \cdot I_{0:t-m}^j}{I_{0:t-m}} \cdot \left(\frac{I_{0:t}^j}{I_{0:t-m}^j} - 1 \right) = \frac{w_b^j}{I_{0:t-m}} \left(I_{0:t}^j - I_{0:t-m}^j \right)$$

Higher level indices

Decomposition, example:

4th quarter=100	Q4	Weights	Q1	Q2	Q3	Q4		Change Q3 to Q4
Elementary index A	100	0,4	102	104	102	104		0,007866
Elementary index B	100	0,3	102	103	104	101		-0,008850
Elementary index C	100	0,3	100	99	99	97		-0,005900
Total index	100	1	101,4	102,2	101,7	101,0		-0,006883

Missing observations and quality adjustment

Missing observations:

a. Temporary: 1. Seasonal goods

2. Other goods

- Missing price is estimated
- Missing price is carried forward
- Variable weights (seasonal goods)

b. Permanently.

Seasonal goods

- Examples:

Price	Period 1	Period 2	Period 3	Period 4	Period 5	Period 6
Good A	10	11	11	12	13	12
Good B	7	8	7,5	8	9	10
Good C	8	8,8	*8,520563*	*9,1913*	*10,14692*	8
Estimation index			96,8	107,9	110,4	
Monthly Index		111,4	96,8	107,9	110,4	93,2
Chained index	100	111,4	107,9	116,4	128,5	119,7

Price	Period 1	Period 2	Period 3	Period 4	Period 5	Period 6
Good A	10	11	11	12	13	12
Good B	7	8	7,5	8	9	10
Good C	8	8,8	*8,8*	*8,8*	*8,8*	8
Monthly Index		111,4	97,9	105,2	106,8	97,7
Chained index	100	111,4	109,0	114,7	122,5	119,7

Missing observations and quality adjustment

Price index measures the prices of a fixed basket.

It is necessary that the sample is continuously updated with new products to ensure representativeness.

To the extent changes in quality is occurring, it is necessary to correct for this so that changes in the price index only reflects "pure" price changes.

Quality adjustment

Time period t-1	Time period t

Quality adjustment

Two types of methods:
-indirect
-explicit

Price	Period 1	Period 2	Period 3	Period 4	Period 5
Good A	350	350	339	350	350
good B	199	199	189	189	189
Good C	429	399			
Good D			499	499	489

Quality adjustment

Direct comparison					
Price	Period 1	Period 2	Period 3	Period 4	Period 5
Good A	350	350	339	350	350
good B	199	199	189	189	189
Good C	429	399			
Good D		*399*	499	499	489
Monthly Index		97,61245	104,7826	101,0701	99,32748
Chained index	100	97,61245	102,2809	103,3754	102,6802

Quality adjustment

Overlapping prices					
Price	Period 1	Period 2	Period 3	Period 4	Period 5
Good A	350	350	339	350	350
good B	199	199	189	189	189
Good C	429	399			
Good D		*499*	499	499	489
Monthly Index		97,61245	97,25534	101,0701	99,32748
Chained index	100	97,61245	94,93332	95,94922	95,30395

Quality adjustment

Bridged overlap					
Price	Period 1	Period 2	Period 3	Period 4	Period 5
Good A	350	350	339	350	350
good B	199	199	189	189	189
Good C	429	399			
Good D			499	499	489
Monthly Index		97,61245	95,91139	101,0701	99,32748
Chained index	100	97,61245	93,62146	94,62332	93,98696

Quality adjustment

Explicit quality adjustment									
Price	**Period 1**	**Period 2**	**Period 3**	**Period 4**	**Period 5**				
Good A	350	350	339	350	350				Kr./GB
good B	199	199	189	189	189		16 GB	399 kr.	24,9375
Good C	429	399					32 GB	499 kr.	15,59375
Good D		798	499	499	489				
Monthly Index		97,61245	83,16604	101,0701	99,32748				
Chained index	100	97,61245	81,1804	82,04913	81,49734				