Interest Rate Risk over the Life-Cycle: A General Equilibrium Approach *

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September 2001

Abstract

This paper examines the consequences of introducing an idiosyncratic uncertain interest rate in a standard life-cycle model à la Auerbach and Kotlikoff (1987). Since the labor market has no uncertainty, labor earnings are used by the consumers to compensate for the risks in the capital market. The multi-period general equilibrium model introduces the possibility for consumers to adjust their labor supply *ex post* in response to new information becoming available (in addition to the opportunity to hedge *ex ante*).

Increased uncertainty causes the number of hours worked to increase, since some old agents start supplying labor to compensate the poor performance of their savings. The framework also makes it possible to quantify the value of labor supply flexibility for these old agents.

JEL classification: D58, D91, H2.

Keywords: idiosyncratic interest rate uncertainty, labor supply flexibility.

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* Thanks to Lars Haagen Pedersen for useful comments and suggestions. Any remaining errors are my responsibility. E-mail: toke.ward.petersen@econ.ku.dk.

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1 Introduction

In a recent article in *Economic Modelling*, Basu, Gosh and Kallianiotis (2001) analyzed the effects of interest rate risk on labor supply and unemployment. The key idea in the paper is that in a basic consumption/savings life-cycle model, interest rate risk will influence the expected benefits from savings. In response the consumer may *ex ante* choose to adjust his (riskless) labor supply to compensate for the risk he faces in the capital markets - in other words labor supply can be used to "hedge" capital earnings.

However, the partial equilibrium analysis by Basu, Gosh and Kallianiotis (2001) is based on a closed-form solution to a simple two-period model, and this places some severe restrictions on the analysis. This paper approaches the problem from a different angle: we use a standard Computable General Equilibrium (CGE) model to investigate the effects of interest rate risk. This approach makes it possible to examine a more realistic model than the two-period model. Often a multi-period problem *can* be reduced to a two-period problem, but it can in this particular case only be done at the expense of important simplifications of the problem.

Below we present an alternative multi-period formulation similar to Auerbach and Kotlikoff (1987), except for the fact that the consumer faces idiosyncratic interest rate uncertainty. The model is used to analyze how increasing uncertainty about the future interest rate affects the economy - in a sense this is a wider question than asked by Basu et al. (2001), that concentrate on the effects on the labor supply and unemployment, and it sheds some light on the issues addressed in Phelps (1962) and Levhari and Srinivasan (1969). The present computations are carried out in a general equilibrium as well as a partial equilibrium framework - this is another novelty compared to the partial equilibrium approach in Basu et al. (2001). In addition our multi-period formulation introduces the opportunity for the consumers to take action *ex post* (i.e. adjust their labor supply after the state of the world is revealed).

With a model specification similar to Auerbach and Kotlikoff (1987) extended with interest rate uncertainty, the simulations show that increased uncertainty means a higher labor supply. The extra hours of labor are supplied by very old consumers who have been "unlucky" with their savings, and who choose to supplement their interest income with income from the labor market. With the two-period model suggested by Basu et al. (2001) these consumers would not have been given this opportunity, since old agents by assumption are excluded from the labor market. The simulations also show that it is not unimportant whether a partial or a general equilibrium model is used. Increasing uncertainty means both lower capital stock and lower labor supply - however the capital stock decreases relatively more than the labor supply, and therefore becomes a relatively scarcer factor. In general equilibrium this influences the factor price ratio, and means relatively higher interest rates - which causes savings to increase relative to the partial equilibrium. Thus the simulations show that a classic two-period partial equilibrium model is not well suited to analyze problems of this kind - the simplifications necessary are not innocent, and more satisfactory results can be obtained with a multi-period computable general equilibrium model.

This paper is organized as follows. Section two describes the model used - a fairly standard general equilibrium model, except for the uncertain interest rate. Section three presents simulations with the model, and section four presents some sensitivity analysis of these results. Finally section five contains a discussion of the results and the implications.

2 Model

The model used by Basu et al. (2001) is as previously mentioned a two-period model. In the first period the consumer, who derives utility from consumption and leisure, chooses how much to work and how much to consume - and saves the residual. In the second period the stochastic interest rate is revealed, and the consumer receives interest income from these savings. By then the consumer is assumed to be retired and cannot supply labor - the only activity for the consumer is to consume the savings plus accrued interest (the latter being stochastic). The stochastic interest rate influences the consumer's labor supply decision in the first period; perhaps he would like to increase his savings for the second period as a buffer against potential low returns on the savings - which would force him to increase his labor supply in the first period. But it could also happen that a consumer faced with a uncertain return on savings will choose to save less, and enjoy consumption with certainty in the first period.

Unfortunately a two-period framework is not well-suited for any realistic analysis of this issue. The main problem is that the consumer, by assumption, cannot supply labor in the last period - and that this last period is half of the model[1]. As pointed out by Bodie, Merton and Samuelson (1992) the flexibility to alter labor supply is valuable to the consumer, and loosing this flexibility makes the consumer overly - and unrealistically - sensitive to changes in the interest rate, since all consumption in the last period is based solely on savings (plus accrued interest) from the first period. When both periods are equally long this overestimates the sensitivity to interest rate uncertainty, since the consumer cannot take any corrective action in the second period, and change his labor supply. In reality, consumers who have a lower-than-expected capital income, can often choose to supply labor, even when they are old. The efficiency of their labor supply may be lower than for young people, but what is important here, is that they have the opportunity to alter their labor supply after the size of the interest rate becomes known. In other words labor supply can be used to smooth earnings even after the realized interest rate is revealed.

In the type of model presented here, where labor can be supplied in every period, labor income can be used to compensate low capital income after the state of the world is revealed (i.e. ex post). This is a different effect (and new effect relative to Basu et al. (2001)) than the hedging consumers perform ex ante, i.e. before the new information becomes available (precautionary behavior).

2.1 The consumer's problem

The economy is populated with overlapping generations of consumers. Consumers live for 55 periods, and face no lifetime uncertainty. The representative consumer has a CES-type life-time utility function

$$U = \frac{1}{1-\frac{1}{\gamma}} \sum_{i=1}^{55} (1+\theta)^{-(i-1)} u_i^{(1-1/\gamma)} \tag{1}$$

where γ is the consumer's intertemporal elasticity of substitution, θ is discount rate, and u_i is an annual utility function. The annual utility function over consumption and leisure is defined by the CES index

$$u_i = \left[c_i^{(1-1/\rho)} + \alpha l_i^{(1-1/\rho)} \right]^{1/(1-1/\rho)} \tag{2}$$

where c_i is consumption in period i, l_i is leisure enjoyed in period i, and where α represents the household's preferences for leisure relative to con-

[1] In the sense that half of the consumer's life-time utility is derived in this period.

sumption, and ρ being the intratemporal elasticity of substitution between leisure and consumption.

The stochastics enter the model through the interest rate on savings. Whereas the interest rate in the Auerbach and Kotlikoff model is assumed to be a constant, r, the interest rate in the present model is stochastic - the realized interest rate for a consumer in period i is r_i. This means that the ex post budget constraint for the consumer is just the discounted stream of future income after taxes minus consumption, which can be written as:

$$\sum_{i=1}^{55} \left[\prod_{j=1}^{i} \frac{1}{(1+r_j)} \right] \left[e_i w (1-l_i)(1-\tau) - c_i \right] \geqq 0 \tag{3}$$

where r_j is the interest rate when the consumer is j years old, w is the standardized wage rate, $(1-l_i)$ is the labor supply, and e_i is the age-dependent productivity profile[2], and τ is the average tax on labor income (that is proportional throughout).

The solution to the consumers problem (maximize (1) subject to the budget constraint (3)) is denoted the consumption level c_j^*, the choice of leisure l_j^*, and the optimal end-of-period asset-holdings a_j^*.[3] Technically the problem is solved using dynamic programming, and the equivalent model formulated as a dynamic programming problem is described in the appendix (here the problem is also described more rigorously).

There is no aggregate uncertainty - the interest rate risk is idiosyncratic. This means that even though the individual consumer does not know his interest rate in the next period, then the distribution of interest rates for the economy as a whole is known (but in equilibrium it is of cause endogenously determined). Uncertainty therefore only exists on the individual and not on the aggregate level - here the law of large numbers apply. However, uncertainty will influence aggregate variables thought its impact on micro behavior.

[2]The age-dependent productivity profile is hump-shaped, and is described in the later sub-section on calibration.

[3]Notice that we have not defined the end-of-period asset-holdings here: it is implicitly defined in the budget constraint (equation 3) and is given by the equation of motion:

$$a_j = (1 + r_d)\, a_{j-1} + w\,(1 - l_j)\, e_j - c_j - \tau[r_d a_{j-1} + w\,(1 - l_j)\, e_j]$$

2.2 The rest of the economy

The production side is identical to Auerbach and Kotlikoff (1987). There is a single good, that is produced using capital and labor subject to a constant-returns-to-scale technology. Production takes place using the CES production function:

$$Y(K,L) = \Lambda \left[\epsilon K^{(1-1/\sigma)} + (1-\epsilon) L^{(1-1/\sigma)}\right]^{1/(1-1/\sigma)} \tag{4}$$

where K and L are capital and labor in the period, Y is output, Λ is a scaling constant, ϵ is a capital-intensity parameter and σ is the elasticity of substitution between K and L.

Since we assume no adjustment costs in K or L, we have the standard result that the gross wages must equal the marginal revenue product of labor (both measured in efficiency units):

$$w = (1-\epsilon)\,\Lambda \left[\epsilon K^{(1-1/\sigma)} + (1-\epsilon) L^{(1-1/\sigma)}\right]^{1/(1-1/\sigma)} L^{-1/\sigma} \tag{5}$$

and the interest rate (in the closed economy) equals the marginal revenue product of capital:

$$r = \epsilon\,\Lambda \left[\epsilon K^{(1-1/\sigma)} + (1-\epsilon) L^{(1-1/\sigma)}\right]^{1/(1-1/\sigma)} K^{-1/\sigma} \tag{6}$$

Notice that the output price is numeraire and there is no depreciation.

The government sector is kept very simple. Government revenue is raised by taxation of labor income, and by taxation of capital income. As in the Auerbach and Kotlikoff (1987) model, government revenue is consumed and not recycled - and this will also be assumed in the base case simulations here[4]. In other words we do not use the assumption that is popular in the public finance literature that the revenue is redistributed back to the consumers.

2.3 Interest rate stochastics

As described above the interest rate is stochastic. As a simplification we assume that the interest rate each period can take one of three values: low,

[4]Notice that utility from government consumption does not enter the utility function directly. However it can be thought of as a component that is additively separable and kept constant (and therefore not modelled explicitly).

medium and high. This variability is not generated from the production side of the economy, but can be thought of as an exogenous disturbance with the following ad-hoc argument. Suppose firms rent the physical capital from the consumers in return of interest payments on the amount of capital borrowed. But (for some unspecified reason - for instance that the checks with payments get mixed up in the mail) these interest payments end up being distributed somewhat random to the consumers: two consumers who both borrow the firm 1 dollar do not necessarily receive the same interest payments. This specification is very ad-hoc, but has one major advantage: there is no need to introduce anything non-standard in the production side of the economy, which makes results easier to understand.

In the default case analyzed below the return in each of the three states are $r(1-\eta), r$ and $r(1+\eta)$ (where r is the deterministic interest rate that enters the producer's problem). If $\eta = 0$ this means that the consumer receives the same interest rate in each state - i.e. there is no uncertainty. If for instance $\eta = 0.1$ this means that a consumer who realizes the low return on his savings get 90% of the interest rate that the consumer in the medium category receives (per unit of capital rented). The missing 10% in this case end up with the lucky consumer who receive an interest rate in the high category. In all simulations in this paper the probability that the consumer ends up in the low category is the same as the probability that the consumer ends up in the high category; this means that the "missing payments" to the consumer in the low category equals what the consumer in the high category receives extra.

In the simulations below the probability of ending in each of the three states is exogenously set to 0.25, 0.50 and 0.25. The probabilities are the same in every period, and independent of the previous period's realization - in other words the stochastic process has no memory (this differs from the autoregressive process used in the previous chapter). The size of η is also set exogenously - the value of η does not influence the expected return on savings, but increasing η implies a mean-preserving increase in variance. In the general equilibrium simulations of cause r, as well as w, are endogenous - and exogenous in the partial equilibrium simulations.

2.4 Calibration

This section describes how the model is calibrated, and how the steady-state is calculated. To get baseline results that are close to Auerbach and Kotlikoff most of their parameters are chosen. The only differences to the A-K set-up is that interest rates differs between consumers..

For the age-dependent productivity, e_i, we use the same equation for productivity over the life-cycle as Auerbach and Kotlikoff (1987) which in turn originate from a cross-sectional regression study by Welch (1979). This hump-shaped profile gives an earnings profile that peaks at age 30, (corresponding to an actual age of 50) at wages that are 45 percent higher than at age 1 (corresponding to 21 years). For the household's intertemporal elasticity of substitution, γ, we use $\gamma = 0.25$, and the one-period discount factor, $\beta = \frac{1}{1.015} \doteq 0.985$. For the taste parameter reflecting the joy of leisure, α, we use Auerbach and Kotlikoff's value of $\alpha = 1.5$, and the elasticity of substitution between leisure and consumption, ρ, is set to 0.8.

Since the production side is identical to Auerbach and Kotlikoff, we use the same parameters as them: the elasticity of substitution: $\sigma = 1.0$ (Cobb-Douglas), the capital intensity parameter: $\epsilon = 0.25$, and the production function constant: $\Lambda = 0.893$. In all simulations in this paper there is a 15% proportional tax on labor and capital income - once again this level of taxation is similar to Auerbach and Kotlikoff (1987).

3 The Effects of an Uncertain Interest Rate

Table 1 on the next page compares various key information for the different economies under consideration; economies that differ only (as far as exogenous variables are concerned) in the size of the interest rate uncertainty: the value of η. A value of $\eta = 1.0$ means that the unlucky consumer in the low category gets no interest payments at all, whereas the lucky consumer (in the high category) receives double interest payments. The table shows the size of production, consumption, leisure, labor supply and capital stock, the utility for a newborn agent as well as the factor price ratio.

partial equilibrium	Standard	η=0.25	η=0.50	η=0.75	η=1.00
Production	100	99.936	99.750	99.436	99.012
Capital stock	100	99.853	99.405	98.634	97.613
Labor supply (units)	100	99.964	99.865	99.705	99.482
Leisure	100	99.986	99.937	99.856	99.748
Avg. Labor eff.	100	99.938	99.747	99.436	99.014
Factor prices (w/r)		- held constant -			
Utility	100	99.985	99.940	99.865	99.762

Table 1. Partial equilibrium (Index: no uncertainty=100 (i.e. $\eta = 0$)).

Note that with the default calibration of the model, suggested by Auerbach and Kotlikoff (1987), the partial equilibrium simulations implies that capital stock goes down as the variability in the interest rate goes up. Leisure goes down, which means that labor supply in hours goes up. At the same time labor supply in efficiency units goes down. This combination is possible because the average labor supply efficiency goes down as well, which will be illustrated in the next section. Thus consumers' utility decreases both because of decreasing leisure and because of the decrease in production (and hence in consumption). Thus overall utility goes down as the variability in the interest rate goes up.

Next consider the case when factor prices are allowed to respond to the decrease in both savings and labor supply in a general equilibrium fashion:

general equilibrium	Standard	η=0.25	η=0.50	η=0.75	η=1.00
Production	100	99.922	99.733	99.447	99.060
Capital stock	100	99.835	99.498	98.908	98.363
Labor supply (units)	100	99.951	99.811	99.575	99.294
Leisure	100	99.994	99.968	99.930	99.853
Avg. Labor eff.	100	99.873	99.685	99.377	98.954
Factor prices (w/r)	100	99.885	99.684	99.332	99.064
Utility	100	99.969	99.894	99.768	99.624

Table 2. General equilibrium (Index: no uncertainty=100).

First notice that the factor price ratio is affected - the relative price on capital goes up (w/r decreases). This is caused by the relatively larger decrease in the capital stock, compared to the size of the decrease in the effective number of units of labor - therefore capital becomes relatively more scarce. This change in the factor price ratio, towards making savings more rewarding, means that the decrease in the capital stock is smaller than in the partial equilibrium framework. Conversely for labor supply: as the relative return to labor declines there is a smaller decrease in leisure, which implies a smaller increase in the number of hours worked. Compared to the partial equilibrium case the net-effect is that the average consumer's utility is lower in the general equilibrium case.

Even though leisure decreases less in general equilibrium (which in isolation would make the consumers better off under general equilibrium), then production and hence consumption is lower under general equilibrium (and this effect would in isolation make the consumers worse off under general equilibrium).

3.1 Life-cycle effects

The simulations give an additional insight, illustrated in Figure 1, that shows the optimal labor supply in hours for the representative consumer:

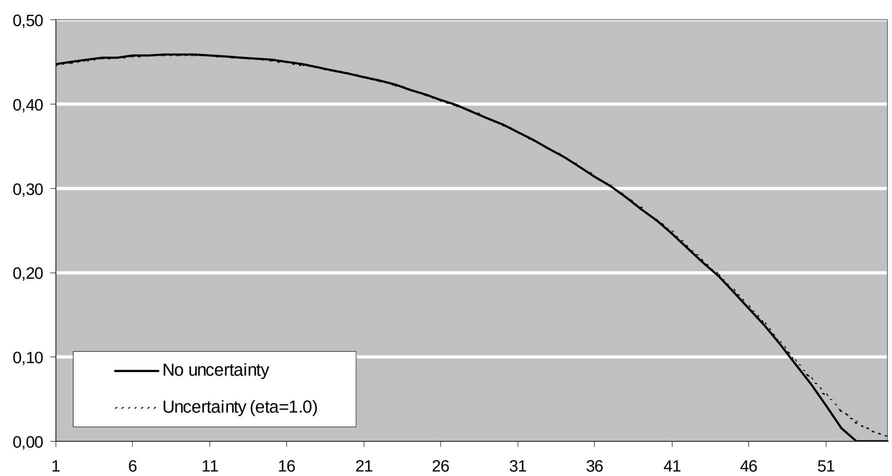

Figure 1: Labor supply when $\eta = 0$ and when $\eta = 1.0$.

The figure shows the from Auerbach and Kotlikoff (1987) well-known hump-shaped labor-supply profile over the life-cycle: when entering the labor market the agents work 45% of their time (which Auerbach and Kotlikoff interpret as 45 hours per week). This number increases over the next couple of years and decline steadily thereafter - and agents retire after 52 years in the labor market. The full line represent the case without uncertainty - corresponding to Auerbach and Kotlikoff. The dotted line shows the general equilibrium model when uncertainty enters the model, i.e. the case where $\eta > 0$. Since the stochastic model does not have one representative consumer, but a large number of different consumers[5], the average value per generation is shown.

The figure explains the puzzling result mentioned above: under uncertainty the labor supply in hours increase, but goes down when measured in efficiency units (i.e. when taking into account that labor at different ages have different productivities). Figure 1 above shows why: on average consumers move their labor supply from the early years to just before their live horizon finish - this represents a move to labor supply when less productive. The figure clearly shows why disallowing labor supply by assumption in a part of the life - as in Basu et al. (2001) - is an assumption that affects the results and that cannot be considered innocent. The simulations show that the optimal choice of consumers on average is to increase their labor supply when old in response to increasing variability in the interest rate - and that one likely introduces a systematic bias when using a two period model where labor supply is disallowed in the last period. One would expect that very rich consumers lowered their labor supply, but the result is not symmetric: rich consumers are (as everyone else) impatient, and try to run their assets down - and are therefore not so rich during the final years.

3.2 The value of labor supply flexibility

As pointed out in the real-options literature à la Dixit and Pindyck (1994) flexibility has value. In this subsection we want to quantify the importance of this labor supply flexibility in a model with a stochastic interest rate. This is done through simulations where consumers are forced into retirement - these simulations can then be compared to simulations where labor supply in the final periods of the consumer's life is unrestricted.

In Auerbach and Kotlikoff's 55-period model without interest rate uncertainty, agents choose to retire from the labor market (i.e. not to supply labor) in the final 3 periods. This is an optimal choice by the consumers (in the absence of uncertainty) and imposing forced retirement in the final 3 periods have no consequences, since this would merely be a non-binding

[5] In the first period everyone are identical since they have not been subject to any uncertainty. When period two begins there are now 3 possible histories for the consumers (they can have received a low, medium or high interest rate), when period three begins there are 9 types of consumers with different histories etc.

constraint. However, as Figure 1 above illustrated this is not the case when the interest rate is uncertain - in this case it is optimal to work for some individuals in the final 3 periods. Imposing a forced retirement on these individuals gives a welfare loss, and the question is how big this loss is.

To investigate this we impose a constraint on the solution to the consumer's problem, that restricts labor supply to zero in the final 3 years. In the benchmark where the interest rate is certain, this has no impact, but will of cause influence the equilibria when the interest rate is stochastic. Table 3 below shows the results of the simulations:

general equilibrium	Standard	η=0.25	η=0.50	η=0.75	η=1.00
Production	100	99.927	99.722	99.392	98.945
Capital stock	100	99.861	99.509	98.956	98.208
Labor supply (units)	100	99.949	99.793	99.538	99.192
Leisure	100	99.997	99.988	99.971	99.944
Avg. Labor eff.	100	99.944	99.770	99.485	99.087
Factor prices (w/r)	100	99.912	99.715	99.414	99.008
Utility	100	99.972	99.898	99.779	99.613

Table 3. General equilibrium with 3 years forced retirement (Index: no uncertainty=100).

The table above should be compared to Table 2 that shows the case without forced retirement. In the presence of the constraint, labor supply in hours increase a lot less, while it decreases more measured in efficiency units. In other words the average efficiency of the labor goes down, but less than was the case without forced retirement; this is because the old (with productivity, e_i, that is lower than average) are prevented from supplying labor. At the same time the capital stock decreases relatively more - in total this means that production decreases more than without the retirement constraint.

Overall utility for an average newborn is - not surprisingly - lower when the consumer is forced to retire. Compared to the unconstrained case it drops 3 percent (not percentage points) when uncertainty increases (η goes from 0 to 1) - this number represents the value of labor supply flexibility for the consumer.

4 Sensitivity Analysis

Obviously the results in the previous sections depend on the specification of the model, as well as the set of parameters used. This section examines how robust the obtained results are to selected changes in the set-up. A thorough sensitivity analysis would require trying many combinations of different parameter values and alternative specifications of the model (for instance the choice of functional forms in the utility function). However, this task is prohibitive, and in line with the standard in CGE-modelling we will restrict the attention a subset of the important parameters.

In this case two elasticities will be analyzed: the intertemporal elasticity of substitution, as well as the elasticity of substitution between leisure and consumption. These two elasticities play an important role for different reasons. The intertemporal elasticity is important in determining how utility at different ages substitute each other. On the other hand the substitutability between leisure and consumption also plays an important role: the easier it is for the consumer to substitutes leisure and consumption (i.e. a high elasticity) the less will variability in either consumption or leisure affect each period's utility.

In the baseline simulations we used Auerbach and Kotlikoff's value for the household's intertemporal elasticity of substitution, $\gamma = 0.25$. Below experiments will be carried out where $\gamma = 0.1$ and $\gamma = 0.5$. In the baseline simulations the elasticity of substitution between leisure and consumption, ρ, was 0.8, and the table below presents experiments with two cases when $\rho = 0.3$ and $\rho = 1.5$. Again this choice of values is the same as the sensitivity analysis in Auerbach and Kotlikoff (1987), and spans quite a wide range for the two parameters, γ and ρ. Table 4 below presents the effects of increasing η from 0 to 1.0, and results are reported as index relative to the situation where $\eta = 0$ (i.e. the value 98.754 in the production column when $\gamma = 0.1$, means that production goes down with 1.246 percent when the variability in the interest rate goes from $\eta = 0$ to $\eta = 1$). The column with the label Default contains the same values as the last column in table 2.

general equilibrium	Default	Alternative specifications			
		$\gamma = 0.1$	$\gamma = 0.5$	$\rho = 0.3$	$\rho = 1.5$
Production	99.060	98.754	99.076	98.935	98.990
Capital stock	98.363	98.598	98.324	98.176	98.359
Labor supply (units)	99.294	98.806	99.328	99.190	99.202
Leisure	99.853	100.009	99.914	99.947	99.970
Avg. Labor eff.	98.954	99.823	99.168	99.109	99.133
Factor prices (w/r)	99.064	99.790	98.990	98.983	99.150
Utility	99.624	98.801	99.915	99.552	99.704

Table 4. Sensitivity analysis for $\eta = 1.00$ (Index: no uncertainty=100).

The table shows that the results are reasonably robust to the relatively large changes in the intertemporal elasticity of substitution and the elasticity of substitution between leisure and consumption. The general results from before hold: interest rate variability makes capital stock as well as labor supply go down. As expected interest rate variability hurts the consumer less when the elasticities of substitution (both γ and ρ) are high: the decrease in utility is lower when $\gamma = 0.5$ or $\rho = 1.5$. When the intertemporal elasticity of substitution is very low ($\gamma = 0.1$), the consumer experiences a drop in welfare of 1.199%, compared a decrease of 0.376% in the standard case - and when the elasticity is very high ($\gamma = 0.5$) the decrease in welfare is only 0.085%.

5 Summary

This paper has examined the effects of introducing an idiosyncratic interest rate risk in a CGE-model. A multi-period model à la Auerbach and Kotlikoff (1987) with idiosyncratic interest rate uncertainty was introduced, and simulations were performed to compare the model's performance with the two-period partial equilibrium used by Basu et al. (2001). The obtained results were also compared with a model where consumers in their final years were forced to retire and not allowed to participate in the labor market.

The simulations showed that increased uncertainty means that the consumers work more hours. This extra labor was primarily supplied by old agents who had been "unlucky" with their savings - and who chose to supplement their capital earnings when old with labor income. This turned out to be an important effect, that by definition is excluded using the two-period framework of Basu et al. (2001); the consumers were given the opportunity to adjust their labor supply *ex post* in response to new information becoming available, as well as the opportunity to hedge *ex ante* (a precautionary motive). This illustrates the point made by Bodie et al. (1992), that flexibility in the labor supply has value to the consumers.

The simulations also show that it is not unimportant whether a partial or a general equilibrium model is used. Increasing uncertainty means both lower capital stock and lower labor supply - however the capital stock decreases relatively more than the labor supply, and therefore becomes a relatively scarcer factor. In general equilibrium this influences the factor price ratio, and means relatively higher interest rates - which causes savings to increase relative to the partial equilibrium. Thus the simulations show that a classic two-period partial equilibrium model is not well suited to analyze the impacts of a stochastic interest rate, and that more realistic results can be obtained with a real life-cycle computable general equilibrium model.

Appendix: Dynamic Programming representation of the consumer's problem

This appendix presents the consumer's problem using dynamic programming - which is the method applied when solving the model using numeric dynamic programming. See Petersen (2001) (as well as Bertsekas (1995) and Ljungqvist and Sargent (2000)) on how to solve dynamic programming problems in practice.

A recursive formulation

In each period the consumer's interest rate in revealed - it can take one of a finite number of values. This set of values is called d, and contains in present simulations three elements: low, medium and high. The recursive maximization problem for a representative consumer with the start-of-period assets a_{j-1}, who in this period will receive interest rate r_d (the category d interest rate) is given by:

$$V_j\left(a_{j-1}, d\right) = \max_{\{c_j, l_j, a_j\}} \left[\frac{1}{(1-1/\gamma)} u\left(c_j, l_j\right)^{(1-1/\gamma)} + \beta \sum_b \pi_b V_{j+1}\left(a_j, b\right) \right] \quad (7)$$

with the budget constraint:

$$a_j = \left(1 + r_d\right) a_{j-1} + w\left(1 - l_j\right) e_j - c_j - \tau\left[r_d a_{j-1} + w\left(1 - l_j\right) e_j\right] \quad (8)$$

where the agent is subject to the liquidity-constraint, the consumption-constraint and the leisure constraint:

$$a_j \geqq 0 \quad (\forall j) \quad (9)$$

$$c_j \geqq 0 \quad (\forall j) \quad (10)$$

$$1 \geqq l_j \geqq 0 \quad (\forall j) \quad (11)$$

where

a_j is the end-of-period assets, e_j is the productivity for an individual j years old, c_j is the consumption in period j, l_j is the leisure enjoyed by generation j, γ is the household's intertemporal elasticity of substitution, β is the one-period discount factor, π_b is the probability that next period's interest rate is in category b [6], r_d is the realization of the interest rate in the current period, w is the wage. Since taxes are proportional τ times current income is the taxes due, and are subtracted.

[6] This implies that $\sum_b \pi_b = 1$.

Solving the consumer's problem

The optimization problem facing an individual is one of finite-state, finite horizon dynamic programming - a Discrete Time Discounted Markov Decision Process. The decision rules can be found by backwards recursion from the last period of life. Since consumers are born without any assets, everyone face an identical problem in the first period of their lives - not until the second period will the return on savings differ between agents.

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