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Abstract

By means of so-called virtual or shadow prices, short-run factor demand functions, short run marginal costs etc. can be derived directly from any long-run cost function. The usual approach (using short-run/restricted/conditional cost functions) is criticized, and some easy approximation formulas are provided. Technical progress, scale effects etc. can be easily added to any cost function by means of factor augmenting efficiency indexes, and it is shown that the trend- and scale parameters of the usual long-run translog cost function are mathematically equivalent to parameters of such efficiency indexes. The techniques are illustrated on the well-known Berndt-Wood data set, using a long-run generalized Leontief (GL) cost function, and assuming capital and labour quasi-fixed.

JEL classification: D21; D24; D45; E23.

Key words: Dynamic factor demand; Cost functions; Shadow/virtual prices; Flexible functional forms; Factor augmenting efficiency indexes.

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[1]This paper derives from the theoretical and practical experience of five years effort to incorporate a consistent factor demand system into the econometric model ADAM. ADAM ("Annual Danish Aggregate Model") is operated by Statistics Denmark, and it is extensively used by Danish government agencies for forecasting and planning purposes. A preliminary draft of the paper has been presented in September 1995 at the Project LINK Fall Meeting in Pretoria, South Africa.

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Contents

1. Introduction

Short-run factor demand functions are very often derived directly (Shephard's Lemma) from a postulated *short-run cost function* (also known as a restricted/-conditional/variable cost function), where the levels of the fixed (or "quasi-fixed") factors appear as independent variables. However, using a short-run cost function as a starting point entails a lot of unpleasant problems, because it is usually very hard to translate relevant economic properties of the underlying production function into corresponding properties of the short-run cost function, especially when operating with more than one quasi-fixed factor. As an example, one might want the underlying production function to be quasi-concave, homothetic, separable, symmetric in the factors or to contain certain kinds of technical progress, and this can be extremely hard and tiresome to translate into properties of the corresponding short-run cost function. In addition to that, short-run cost functions have a tendency to imply unreasonable isoquants of the underlying production function, as shown in section 4 below.

To avoid this problem, one might alternatively base one's empirical work on a postulated *long-run cost function*, since there are generally quite easy and straightforward correspondences between properties of the underlying production function and properties of the long-run cost function. Long-run factor demands can be derived directly (Shephard's Lemma) from the long-run cost function, so the great question − and the great question of this paper − is how to get from these concepts to the corresponding *short-run* concepts, where the levels of the quasi-fixed factors appear as independent variables? How does one get from a long-run cost function to its corresponding short-run counterpart, when the underlying production function is not known? Or equivalently: how does one get from a system of long-run factor demand functions to the corresponding system of short-run factor demand functions?

Fortunately, this question is easy to answer, as there exists a very useful − but often overlooked − *direct link* between any long-run factor demand system and its corresponding short-run factor demand system, making use of so-called shadow or virtual prices of the quasi-fixed factors (see e.g. Rothbart (1941), Neary/Roberts (1980), Squires (1994)). This technique is easy: one simply solves the long run demand functions for the quasi-fixed factors with respect to their own factor prices and inserts the resulting shadow or virtual prices into the long run demand functions for the remaining (flexible/variable) factors. The result of this is the corresponding system of short-run factor demand functions.

Using this technique, one can form theoretically consistent short- and long-run factor demand functions in three easy steps: (a) postulate a long-run cost function (with the desired theoretical properties imposed), (b) derive the corresponding long-run factor demands using Shephard's Lemma, and finally (c) derive the corresponding *short-run* factor demands from the long-run factor demands by the above-mentioned shadow price method. From this, short-run costs follow directly, and short-run marginal costs can be obtained from long-run marginal costs in an equally simple way.

Thus, the above shadow price link between short- and long-run factor demand makes it possible to go from a long-run cost function to all necessary short-run concepts in an easy and very transparent way. But as this approach does not seem to be widely utilized, there has been an evolution of a large number of different *short-run* cost functions (for recent examples, see e.g. the short-run translog of Berndt/Friedlaender/Chiang/Vellturo (1993) or the short-run generalized Leontief (GL) of Park/Kwon (1995)). These functions usually resemble − but are in reality very different from − well-known cost functions such as the (original) long-run translog or generalized Leontief cost functions. As mentioned above, it is not exactly easy to impose theoretical restrictions on a short-run cost function, and in addition to that, the proposed alternative shadow price approach − holding on to well-known and well-examined long-run cost functions − is much more simple and straightforward, both as to the mathematical derivations and as to the conceptual clarity.

As regards technical progress, scale effects etc., it is shown how to introduce these effects in a very easy and easily interpretable way, making use of so-called (factor augmenting) *efficiency indexes*. Using this idea, one can start out with a "raw" no technical progress/constant returns to scale long-run cost function, and add technical progress, scale effects etc. via these efficiency indexes. Most interestingly, it can be shown that all the trend- and scale parameters of the usual translog cost function can be exactly translated into corresponding efficiency parameters, yielding a completely new way of interpreting the trend- and scale parameters of the translog cost function.

In section 2 below, the two above-mentioned cost function approaches are contrasted. In section 2A, the usual *short-run cost function* approach is examined, whereas the proposed alternative shadow price approach is presented in section 2B, using a *long-run cost function* as the starting point instead. The two different approaches are illustrated using the generalized Leontief cost function in a short- and long-run version. In section 3, the results of section 2 are summarized, and the shadow price approach is sketched in the general *n*-factor case. In section 4, the theoretical problems with the usual short-run cost function approach are demonstrated, since the isoquants of the underlying production function often turn out to have a very unattractive shape. In section 5 it is shown in the *n*-factor case how to form all relevant short-run concepts from the original long-run generalized Leontief cost function, generalizing the results of section 2B. In section 6, the above-mentioned efficiency index methodology is presented, and in section 7, some very useful general approximations between short- and long-run factor demand are provided.

In section 8, a simple empirical application on the well-known Berndt-Wood data set is briefly presented, and in section 9, the paper is concluded.

2. Two different approaches to short- and long-run factor demand

As mentioned in the introduction, the main point of this paper is that it is possible to form short-run factor demands (and costs and marginal costs) directly from a long-run cost function. This entails many advantages, since the global properties of the most popular (flexible) long-run cost functions such as e.g. the translog and the generalized Leontief have been analyzed in great depth in numerous papers (see e.g. Caves/Christensen (1980), Barnett/Lee (1985), Despotakis (1986), Diewert/Wales (1987)). This is in sharp contrast to the absence of similar studies of short-run cost functions, despite the fact that the latter have been used repeatedly in the literature.

In the following two sections (2A and 2B), the above claims are illustrated using a particular cost function − the so-called generalized Leontief (see Diewert (1971)) − in a short- and long-run version. And to keep matters as simple as possible − but still being able to focus on the main point of this paper − it is assumed that there is only three production factors (one quasi-fixed, and two flexible), no technical progress and constant returns to scale. In section 2A below, the conventional short-run cost function approach is examined following the usual procedure closely, and making no use of the shadow price results presented in section 2B.

2A. The usual short-run cost function approach

The short-run cost function approach was initiated by Brown and Christensen (1981), and has been used innumerable times since then. For a recent use of a short-run translog cost function, see e.g. Berndt/Friedlaender/Chiang/Vellturo (1993), and for a recent use of a short-run generalized Leontief cost function, see e.g. Park/Kwon (1995), building on Morrison (1988). This particular short-run version of the generalized Leontief − being perfectly representative of short-run cost functions in general, including the short-run translog − is presented below. Subsequently, this function is denoted the "short-run Morrison generalized Leontief cost function", and with three production factors (one quasi-fixed and two flexible), no technical progress and constant returns to scale, total short-run costs, *C*, are given as:[3]

[3]The generalization of the original GL function to contain among other things the quasi-fixed factors is not very unequivocal, since the way the quasi-fixed factors are added to the GL function differs among writers on the subject. The Morrison approach, however, is fully capable of illuminating

$$C \;=\; P_K K \;+\; Y\left[\sum_i\sum_j \;\alpha_{ij}\; P_i^{0.5}P_j^{0.5} \;+\; \left(\frac{K}{Y}\right)^{0.5}(\delta_{KL}P_L+\delta_{KE}P_E)\right.$$

$$\left. +\;\gamma_{KK}\left(\frac{K}{Y}\right)\;(P_L+P_E)\right]\;, \qquad \textbf{(1)}$$

where i, j = L, E and $\alpha_{LE} = \alpha_{EL}$. The variables K, L and E, denote fixed capital, labour and energy, P_K, P_L and P_E, are the corresponding factor prices, and Y is the production level. The function is homogenous of degree one in the factor prices, and it is seen that there is constant returns to scale, since C increases with 1%, if Y and K are both increased by 1%. The first term of (1) is simply the fixed costs of K, whereas the second term is a generalized Leontief form in the prices of the two flexible factors (L and E). The last two terms express how the variable costs are affected by K, and the concrete mathematical form contains 6 parameters necessary for flexibility (cf. Lau (1974) regarding the definition of flexibility), but is otherwise quite arbitrary. Differentiating (1) with respect to P_L and P_E (Shephard's Lemma) yields the short-run factor demands:

$$L \;=\; Y\left[\alpha_{LL} \;+\; \alpha_{LE}\left(\frac{P_E}{P_L}\right)^{0.5} \;+\; \delta_{KL}\left(\frac{K}{Y}\right)^{0.5} \;+\; \gamma_{KK}\,\frac{K}{Y}\right]\;, \qquad \textbf{(2)}$$

$$E \;=\; Y\left[\alpha_{EE} \;+\; \alpha_{LE}\left(\frac{P_L}{P_E}\right)^{0.5} \;+\; \delta_{KE}\left(\frac{K}{Y}\right)^{0.5} \;+\; \gamma_{KK}\,\frac{K}{Y}\right]\;, \qquad \textbf{(3)}$$

whereas the short-run marginal costs, $MC \equiv \partial C/\partial Y$, are obtained by differentiating C with respect to Y:

$$MC \;=\; \sum_i\sum_j \;\alpha_{ij}\; P_i^{0.5}P_j^{0.5} \;+\; 0.5\left(\frac{K}{Y}\right)^{0.5}\;(\delta_{KL}P_L+\delta_{KE}P_E)\;,\quad i,j \;=\; L,E\,. \qquad \textbf{(4)}$$

At this point, the users of the short-run cost function approach usually define a so-called shadow price of K, $\tilde{P}_K$, denoting by how much the short-run variable costs

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the main points. The function is taken from Morrison (1988), equation (1), with $\delta_{im} = \gamma_{mn} = \gamma_{mk} = 0$ (or Park/Kwon (1995) equation (12) with the same restrictions). Please note that I have added the fixed costs, $P_K K$, on the right hand side, so that (1) is the short run *total* costs, whereas the Morrison's equation yields the short run *variable* costs. This is just for convenience and ease of comparison.

are reduced, when K is increased by one unit. Short-run variable costs are given as $G = P_L L + P_E E = C - P_K K$, so that $\tilde{P}_K = -\partial G/\partial K = P_K - \partial C/\partial K$, which in this case is:

$$\tilde{P}_K = -0.5 \left(\frac{K}{Y}\right)^{-0.5} (\delta_{KL} P_L + \delta_{KE} P_E) - \gamma_{KK} (P_L + P_E) \ . \qquad \textbf{(5)}$$

The long-run equilibrium condition implies that $\tilde{P}_K = P_K$ (being equivalent to $\partial C/\partial K = 0$; that is, minimizing the total short-run costs with respect to K). This condition yields the long-run stock of capital, K^*, as:

$$K^* = 0.25\, Y \left(\frac{-\delta_{KL} P_L - \delta_{KE} P_E}{P_K + \gamma_{KK} (P_L + P_M)} \right)^2 . \qquad \textbf{(6)}$$

If K^* is inserted into the short-run demands for the flexible factors (L and E), the long-run demands for the flexible factors (L^* and E^*) are obtained:[4]

$$L^* = Y \left[\alpha_{LL} + \alpha_{LE} \left(\frac{P_E}{P_L}\right)^{0.5} + 0.50\, \delta_{KL} \left(\frac{-\delta_{KL} P_L - \delta_{KE} P_E}{P_K + \gamma_{KK} (P_L + P_M)} \right) \right.$$

$$\left. + 0.25\, \gamma_{KK} \left(\frac{-\delta_{KL} P_L - \delta_{KE} P_E}{P_K + \gamma_{KK} (P_L + P_M)} \right)^2 \right] , \qquad \textbf{(7)}$$

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[4]The regularity conditions $\partial^2 C/\partial K^2 > 0$ and $\partial K^*/\partial P_K < 0$ together imply that $-\delta_{KL} P_L - \delta_{KE} P_E > 0$ and $P_K - \gamma_{KK}(P_L + P_M) > 0$ so that the numerator and denominator of the fractions are both positive. Therefore, no numerical sign is needed in the first fraction in (7) and (8), provided that the above conditions are observed. For a description of regularity conditions for short-run cost functions, see Browning (1983).

$$E^{*} = Y\left[\alpha_{EE} + \alpha_{LE}\left(\frac{P_L}{P_E}\right)^{0.5} + 0.50\,\delta_{KE}\left(\frac{-\delta_{KL}P_L - \delta_{KE}P_E}{P_K + \gamma_{KK}(P_L+P_M)}\right)\right.$$

$$\left. + 0.25\,\gamma_{KK}\left(\frac{-\delta_{KL}P_L - \delta_{KE}P_E}{P_K + \gamma_{KK}(P_L+P_M)}\right)^2\right] . \quad \textbf{(8)}$$

If K^* is inserted into the short-run cost function (C), the long-run cost function (C^*) results:

$$C^{*} = Y\sum_i\sum_j \alpha_{ij}\,P_i^{0.5}P_j^{0.5} - 0.25\;Y\,\frac{(-\delta_{KL}P_L - \delta_{KE}P_E)^2}{P_K + \gamma_{KK}(P_L+P_E)}, \quad i,j = L,E . \quad \textbf{(9)}$$

Finally, long-run marginal costs, $MC^* \equiv \partial C^*/\partial Y$, can be calculated as

$$MC^{*} = \sum_i\sum_j \alpha_{ij}P_i^{0.5}P_j^{0.5} - 0.25\,\frac{(-\delta_{KL}P_L - \delta_{KE}P_E)^2}{P_K + \gamma_{KK}(P_L+P_E)} . \quad \textbf{(10)}$$

2B. The alternative long-run cost function approach (shadow/virtual prices)

In this section, the above-examined short-run cost function approach is contrasted with the proposed long-run cost function approach, basing the derivations on a *long-run* cost function instead − in this particular example the (original) long-run generalized Leontief due to Diewert (1971) − and using shadow or virtual prices to obtain corresponding short-run factor demands, costs and marginal costs.

Doing this, it is possible to turn the procedure of section 2A upside down, beginning with a long-run cost function and ending up with a short-run cost function. Operating as previously with three production factors, no technical progress and constant returns to scale, the long-run costs are given as (see Diewert (1971)):

$$C^{*} = Y\sum_i\sum_j \beta_{ij}\,P_i^{0.5}\,P_j^{0.5}, \qquad i,j = K,L,E , \qquad \beta_{ij} = \beta_{ji} . \quad \textbf{(11)}$$

The long-run generalized Leontief is homogenous of degree one in the factor prices, and the long-run factor demands follow from Shephard’s Lemma:

$$K^* = Y\left[\beta_{KK} + \left(\beta_{KL}\, P_L^{0.5} + \beta_{KE}\, P_E^{0.5}\right) P_K^{-0.5}\right], \tag{12}$$

$$L^* = Y\left[\beta_{LL} + \left(\beta_{KL}\, P_K^{0.5} + \beta_{LE}\, P_E^{0.5}\right) P_L^{-0.5}\right], \tag{13}$$

$$E^* = Y\left[\beta_{EE} + \left(\beta_{KE}\, P_K^{0.5} + \beta_{LE}\, P_L^{0.5}\right) P_E^{-0.5}\right]. \tag{14}$$

Finally, the long-run marginal costs are obtained by differentiating (11) with respect to Y:

$$MC^* = \sum_i \sum_j \beta_{ij}\, P_i^{0.5}\, P_j^{0.5}, \qquad i,j = K, L, E . \tag{15}$$

At this point, the crucial question − and the crucial question of this paper − is how to get from the long-run cost function (or factor demands) to the corresponding system of short-run factor demands, where K is fixed at a predetermined level. At first sight this seems impossible, as K does not appear in the long-run cost function − and since the underlying production function is not known. But fortunately the answer is very simple, as the problem is solved simply by artificially altering the price of K, P_K, until K^* in (12) is equal to the predetermined level K. When the long-run demands for the flexible factors, L^* and E^*, are evaluated at this artificial price, they yield the short-run demands for the self-same factors (see e.g. Neary/Roberts (1980) or Squires (1994) for proofs and details. See also Pollak (1969) and Deaton (1986) regarding rationing and shadow prices).

In the literature on rationing in demand systems, this price is usually denoted a "virtual" price after Rothbart (1941), whereas it is more obvious to denote the artificial price a shadow price in the context of producer behavior. This is so, because it turns out that the shadow/virtual price concept defined here coincides with the shadow price concept defined in the preceding section; that is, defined as $\tilde{P}_K = -\partial G/\partial K = P_K - \partial C/\partial K$; or minus the short-run variable costs differentiated with respect to K (see Squires (1994) p. 238 for the proof).

The shadow price of the quasi-fixed factor, $\tilde{P}_K$, is consequently found by solving K^* (12) with respect to its own price, P_K. This yields the following expression:

$$\tilde{P}_K = \left(\frac{\beta_{KL} P_L + \beta_{KE} P_E}{K/Y - \beta_{KK}}\right)^2 . \tag{16}$$

Substituting $\tilde{P}_K$ for P_K in the long-run factor demands for the flexible factors ((13)-(14)), the following equations are obtained:

$$L = Y\left[\beta_{LL} + \left(\beta_{KL}\frac{\beta_{KL}P_L^{0.5} + \beta_{KE}P_E^{0.5}}{K/Y - \beta_{KK}} + \beta_{LE}P_E^{0.5}\right)P_L^{-0.5}\right], \tag{17}$$

$$E = Y\left[\beta_{EE} + \left(\beta_{KE}\frac{\beta_{KL}P_L^{0.5} + \beta_{KE}P_E^{0.5}}{K/Y - \beta_{KK}} + \beta_{LE}P_L^{0.5}\right)P_E^{-0.5}\right]. \tag{18}$$

In this particular factor demand system, it is seen that K enters the demand for L and E through the denominator ($K/Y - \beta_{KK}$), pushing the short-run demand for L and E towards infinity, as the capital-output ratio, K/Y, is lowered towards its limit, β_{KK} (assuming here that β_{KK} is positive). Here, the parameter β_{KK} is crucial for the size of the physical capacity limit, since K cannot be lowered below $\beta_{KK}Y$ for given Y, and Y cannot exceed K/β_{KK} for given K.

The corresponding short-run cost function can be found by means of a central formula linking short- and long run costs together: $C = C^*(Y, \tilde{P}_K, P_L, P_E) + (P_K - \tilde{P}_K)K$, that is, evaluating the long-run cost function $C^*(\cdot)$ at the shadow price of K, and adding the difference between the observed and the shadow price of K multiplied by K itself (see e.g. Squires (1994) p. 238). Alternatively, it can of course be found simply by inserting L and E into the definition of the short-run cost, $C = P_K K + P_L L + P_E E$. Both methods yield:

$$C = P_K K + Y\sum_i\sum_j \beta_{ij}P_i^{0.5}P_j^{0.5} + Y\frac{\left(\beta_{KL}P_L^{0.5} + \beta_{KE}P_E^{0.5}\right)^2}{K/Y - \beta_{KK}}, \; i,j = L, E. \tag{19}$$

Regarding short-run marginal costs, the shadow price, $\tilde{P}_K$, can be used once again, as it can be shown that substituting $\tilde{P}_K$ for P_K in the expressions for the long-run marginal costs, MC^*, gives the short-run marginal costs, MC (see appendix A for the proof). Alternatively, MC could be found by differentiating (19) with respect to Y.

$$MC = \sum_i \sum_j \beta_{ij} P_i^{0.5} P_j^{0.5} + (2K/Y - \beta_{KK}) \left(\frac{\beta_{KL} P_L^{0.5} + \beta_{KE} P_E^{0.5}}{K/Y - \beta_{KK}} \right)^2, \quad i = L,E \ . \tag{20}$$

3. Advantages of the long-run cost function approach (shadow/-virtual prices)

In section 2A it was shown how to derive long-run factor demands (and costs) from a short-run cost function, and in section 2B, the opposite was shown: namely how to form short-run factor demands (and costs) from a long-run cost function. All in all, four cost functions have been shown, as illustrated below:

Table 1. The cost functions of section 2A and 2B

Approach	Section	Short run		Long run
The short-run cost function approach	2A	Equation (1) Morrison (1988)	→	Equation (9) Follows from (1)
The long-run cost function approach	2B	Equation (19) Follows from (11)	←	Equation (11) Diewert (1971)

Note: the arrows indicate that one of the cost functions is used as basis, from which the other one is derived.

The short-run Morrison GL cost function (1) and the long-run Diewert GL cost function (11) are the ones traditionally presented and used in the literature, and are fully representative of short- and long-run cost functions in general. The *long-run* Morrison GL cost function ((9), derived from (1)) is not usually brought to light, and the *short-run* Diewert GL cost function ((19), derived from (11)) has not to my knowledge been presented before.

Regarding the long-run cost functions of section 2A and 2B, it is noticed that the long-run Morrison and Diewert cost functions are different mathematical expressions with different properties, as is seen by comparing equations (9) and (11). Actually these forms do not have much in common, apart from the quadratic form of the prices of L and E. Comparing alternatively the short-run cost functions (equations (1) and (19)), these are necessarily different as well.[5]

Now, the whole point of this paper is to claim that it would be much simpler and better to use a well-known long-run cost function such as e.g. the long-run Diewert generalized Leontief as a basis, instead of trying to create a short-run cost function oneself, such as the case is with e.g. the Morrison GL cost function. This is so, because it is much easier to start out with a long-run cost function, but also because the global properties of the well-known cost functions are already analyzed in great detail (Caves/Christensen (1980), Barnett/Lee (1985), Despotakis (1986), Diewert/Wales (1987)).

In the following, the principles of the shadow price approach are summarized, this time in the general case with n factors, of which k are quasi-fixed and $l = n - k$ are flexible. The approach is illustrated in table 1, explanations are below.

Table 2. Overview of the shadow price approach (of section 2B)

	Long run			Short run		
Demand for quasi-fixed factors	(a1)	$\boldsymbol{X_k}^*$	$= \mathrm{X}_k^*(Y, \boldsymbol{P_k}, \boldsymbol{P_l})$	(b1)	$\boldsymbol{X_k}$	$= \mathrm{X}_k^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P_l})$
Demand for flexible factors	(a2)	$\boldsymbol{X_l}^*$	$= \mathrm{X}_l^*(Y, \boldsymbol{P_k}, \boldsymbol{P_l})$	(b2)	$\boldsymbol{X_l}$	$= \mathrm{X}_l^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P_l})$
Marginal costs	(a3)	MC^*	$= \mathrm{MC}^*(Y, \boldsymbol{P_k}, \boldsymbol{P_l})$	(b3)	MC	$= \mathrm{MC}^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P_l})$

Note: Exogenous variables: Y, $\boldsymbol{P_k}$, $\boldsymbol{P_l}$ and $\boldsymbol{X_k}$. Endogenous variables: $\boldsymbol{X_k}^*$, $\boldsymbol{X_l}^*$, MC^*, $\tilde{\boldsymbol{P}}_k$, $\boldsymbol{X_l}$ and MC.

In the first column, the long-run factor demand functions and marginal cost function are presented. These are often derived from a cost function, but could just as well originate from a production function. The variables are to be interpreted as follows: $\boldsymbol{X_k}$ is a vector of the k quasi-fixed factors, and $\boldsymbol{X_l}$ is a vector of the $l = n - k$ flexible factors. The variables $\boldsymbol{P_k}$ and $\boldsymbol{P_l}$ are vectors of the corresponding factor prices. Given the functional forms of $\mathrm{X}_k^*(\cdot)$, $\mathrm{X}_l^*(\cdot)$ and $\mathrm{MC}^*(\cdot)$, the calculation of $\boldsymbol{X_k}^*$, $\boldsymbol{X_l}^*$ and MC^* is straightforward, as Y, $\boldsymbol{P_k}$ and $\boldsymbol{P_l}$ are exogenous variables.

Turning to the short-run behavior, the first step is to find the k shadow prices, $\tilde{\boldsymbol{P}}_k$, ensuring that (b1) in table 2 is observed − assuming here, that these shadow prices exist and are unique. With these k shadow prices, the calculation of $\boldsymbol{X_l}$ and MC is

[5]Using the derived short-run Diewert GL cost function (19) as a basis, one could actually deduce all the equations of section B by using the procedure outlined in section A, and one could also take the derived long run Morrison GL cost function (9) as a basis and derive all the equations of section A by the means presented in section B.

straightforward, since the functional forms, $X_l^*(\cdot)$ and $MC^*(\cdot)$ are simply reused. If (b1) is not analytically solvable with respect to the factor prices, the solution might be found numerically. Alternatively, the approximations using long-run partial price elasticities shown in section 7 could be very useful. If one does not already know the long-run cost function, it is given as $C^* = C^*(Y, \boldsymbol{P_k}, \boldsymbol{P_l}) = \boldsymbol{P_k}'\boldsymbol{X_k}^* + \boldsymbol{P_l}'\boldsymbol{X_l}^*$, whereas short-run costs can be found by using the formula $C = C^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P_l}) + (\boldsymbol{P_k}-\tilde{\boldsymbol{P}}_k)'\boldsymbol{X_k}$; that is, evaluating the long run cost function at the k prices of the quasi-fixed factors and adding the difference between the observed and shadow prices multiplied by the factors themselves. This relationship is quite obvious, as $C^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P_l}) = \tilde{\boldsymbol{P}}_k'\boldsymbol{X_k} + \boldsymbol{P_l}'\boldsymbol{X_l}$ (see e.g. Squires (1994) p. 238). Alternatively, short-run costs are of course obtainable as $C = \boldsymbol{P_k}'\boldsymbol{X_k} + \boldsymbol{P_l}'\boldsymbol{X_l}$. Total short-run costs differentiated with respect to the k quasi-fixed factors are simply given as $\partial C/\partial \boldsymbol{X_k} = \boldsymbol{P_k}-\tilde{\boldsymbol{P}}_k$ (see e.g. Squires (1994) p. 238).

To sum up so far, the advantage of the shadow price approach sketched in table 2 is that once the nut is cracked (that is, the shadow prices of the quasi-fixed factors are found in (b1)), there is no need for further nut-cracking, as the rest of the calculations − given these shadow prices − only involves reusing the long-run equations. In addition to that − as it will be demonstrated in the next section − the short-run cost function approach almost inevitably implies serious asymmetries of the isoquants of the underlying production function, potentially inducing substantial problems in applied work.

4. Problems with the short-run cost function approach

In the last section, I suggested using shadow prices to obtain the corresponding short-run cost function and factor demands from one of the existing and well-known long-run cost functions. As previously mentioned, this has many advantages, as all the well known mathematical expressions regarding the long-run cost function − such as long-run elasticities, regular regions (consistency domains), separability restrictions etc. − can be reused. Apart from these conveniences, however, I wish to draw attention to a potentially very serious problem with the usual short-run cost functions, as they may imply quite unrealistic long-run factor demand functions.

This problem is illustrated by means of the short-run Morrison GL cost function of section 2A, comparing it with the long-run Diewert GL cost function of section 2B. Regarding the short-run cost function approach, it will be shown that when the substitution between the quasi-fixed and the other factors is low, the isoquants of the underlying production function become unreasonable – without being out of the consistency domain (regular region). Actually, one could say that the short-run Morrison generalized Leontief cost function has a misleading name, as the functional form behaves very strangely, when the substitution between K and the other factors is sufficiently low (that is, approaching the Leontief special case).

To illustrate this, I have chosen the six parameters of each of the two cost functions so that they both yield $K^* = L^* = E^* = 1$ at prices $P_K = P_L = P_E = 1$ and production level $Y = 1$. In addition to that, the parameters are chosen so that the long-run partial price elasticities of the two functions are equal in the considered point, as all own-price partial price elasticities are set equal to −0.33, and all cross-price partial elasticities are set equal to 0.17. This implies that all Allen-Uzawa elasticities of substitution (AUES) are 0.50 − that is, halfway between the Leontief (AEUS = 0) and the Cobb-Douglas (AUES = 1) special cases. The interesting issue is how the functions behave away from the selected point.[6]

By altering the factor prices, the underlying isoquant may be drawn using the long-run factor demand functions ((6)-(8) and (12)-(14), respectively), as it is done in the following 3-D graphs. In the graphs, origo is in the most distant bottom right-hand corner, and three lines intersect at the central point, (1, 1, 1), showing how the factor demand system responds to modifications in one of the three factor prices respectively. The circular rings denote combinations of the factor prices, where the relative prices are twisted by 20%, 44% and so on, so that e.g. the six points of intersection between the innermost circle and the lines show points on the isoquant where one of the three factor prices is increased or decreased with 20% relative to the others. The maximal twist in the relative prices is a factor 5.16, which is seen as the ninth and outermost "circle".[7]

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[6]In the considered "point", all factors are net substitutes, and all cost shares are equal, whereas the substitution possibilities are in the low end of the spectrum. On the other hand, own-price partial elasticities of substitution of −0.33 are not at all implausible, and elasticities of that magnitude can be found in Morrison's own empirical work (Morrison (1988)), or in section 8 of this paper. Here, the parameters in the Morrison GL are $\alpha_{LL} = \alpha_{EE} = 4$, $\alpha_{LE} = 0.5$, $\delta_{KL} = \delta_{KE} = -6$ and $\gamma_{KK} = 2.5$. In the Diewert GL all the β_{ij}'s are equal to 1/3.

[7]Technical note: the circles in the 3-D graphs are constructed by making the three factor prices run through the values $\ln(P_i) = 2/3 \ln(R) \sin(x + 2/3\, \pi\, (i-1))$, $i = 1, 2, 3$ and $0 \le x < 2\pi$. The parameter R indicates how much the relative prices are twisted, and in the graphs R = 1, 1.20, 1.44, ... , 5.16, respectively. For each choice of R, the above construction contains the six special cases where one of the three factor prices is increased or decreased by a factor R relative to the others (these six cases occurring at $x = \pi/6$, $3\pi/6$, $5\pi/6$ and so forth). This method is quite useful for depicting underlying isoquants of a three-factor long-run factor demand system.

Figure 1. Isoquant for the Morrison short-run GL cost function

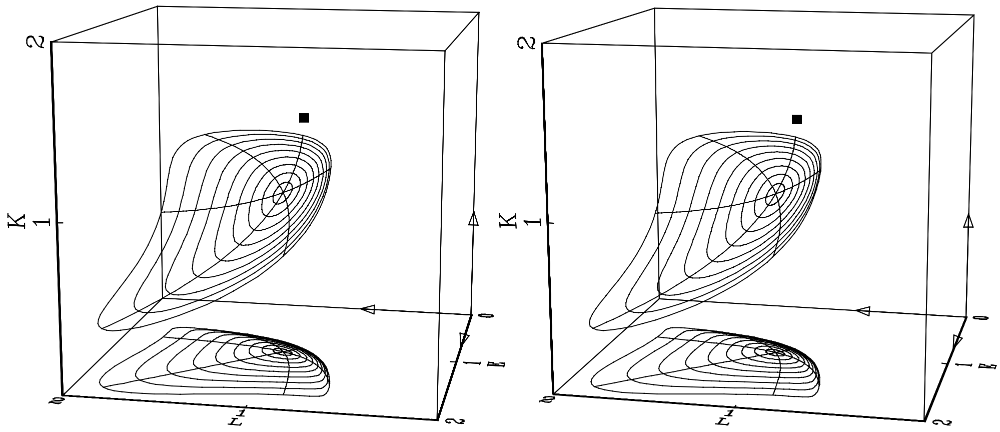
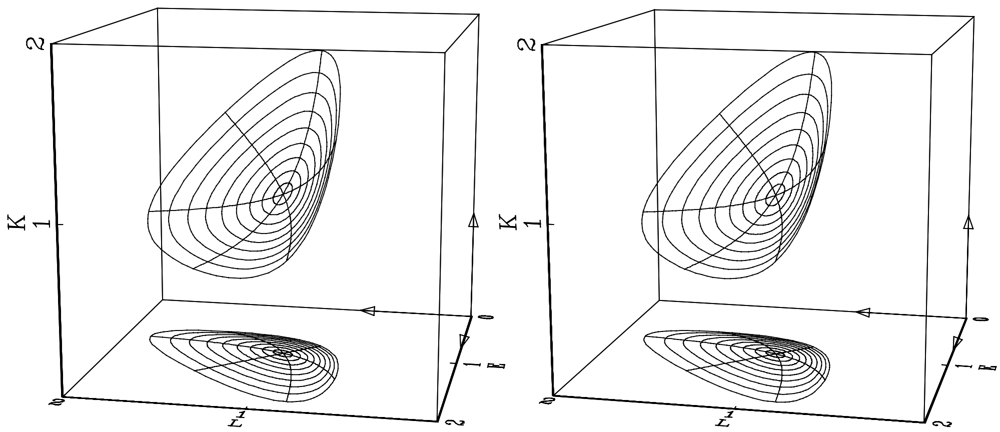


Figure 2. Isoquant for the Diewert long-run GL cost function



Note: the above graphs are pairwise stereograms, so that a 3-D image may be obtained, when the left and the right images are made overlap by the eyes. This is most easily done holding the page in outstretched arms. The pictures at the bottom of each box are 2-D "shadows" of the above figures.

In figure 1, the line from the center of the figure in the direction of the black square indicates how the three factors respond to a reduction in the price of *K*. This increases the demand for *K* and decreases the demand for both *L* and *E*. The main problem of the short-run cost function is that when the price of *K* is diminished towards zero, the demand for all three factors tends towards a particular positive level (marked by the black square), in contrast to the long-run cost function (see figure 2), where K tends to infinity.[8] Increasing P_K in the short-run cost function in figure 1 means following the above-mentioned line in the reverse direction, and it is seen that increasing P_K by a factor five gives dramatic increases in the demand for L and E, compared to the long-run cost function.[9]

What is particularly unpleasant about this is that the short-run GL cost function could not be rejected on account of not being theoretically consistent with the neoclassical assumptions. Actually, the isoquant in figure 1 (and 2 as well) is globally concave and does not yield negative factor demands anywhere. This means that the concept of "well-behavedness" or global consistency could perhaps need to be tightened, since it does not e.g. rule out isoquants implying that all demands converge to a specific strictly positive level (the black square in figure 1), when a chosen factor price (here: P_K) is driven towards zero. With the short-run GL cost function this problem becomes worse, the smaller the substitution between K and the other factors is. In this respect, as mentioned above, the short-run GL does not fully live up to its name (containing the Leontief case as a special case), as very unpleasant implications result from the substitution between K and the other factors being close to zero.[10]

Finally, it should be emphasized that the usual formulation of the short-run translog cost function (for the first example of using this form, see Atkinson/Halvorsen (1976); for recent examples, see e.g. Berndt/Friedlaender/Chiang/Vellturo (1993), Nemoto/Nakanishi/Madono (1993), or Shah (1992)) suffers from exactly the same problem. Such a short-run translog − where K is introduced in the quadratic form in the same way as the factor prices − has the further disadvantage that it is not possible to solve the equation yielding K^* analytically (in closed form), complicating the analysis considerably.

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[8]The reason why the demand for the three factors tend to a particular point (the black square) can be seen in the short-run demands for the flexible factors (eq. (2) and (3)). In this concrete example, δ_{KL} and δ_{KE} are both negative, whereas γ_{KK} is positive, implying that if K is made sufficiently large, $\partial L/\partial K$ and $\partial E/\partial K$ (for given Y) will sooner or later *both* become positive, implying that the isoquant is bending in an inadmissible way. The black square is in fact denoting a limiting case, where $\partial L/\partial K = \partial E/\partial K = 0$.

[9]Of course, nobody says that the isoquant of the long-run Diewert GL cost function is the truth, but actually this isoquant and a three-factor CES-isoquant with elasticity of substitution $\sigma = 0.50$ and equal cost shares turn out to be identical. So the conclusion is that the short-run Morrison GL cost function differs much from a globally well-behaved functional form such as the CES − at least in this (not unreasonable) case given the chosen elasticities and cost shares. Apart from that, one can argue that there could be no *a priori* reason for treating one of the factors asymmetrically, as the case is with K in the short-run Morrison GL (and short-run cost functions in general).

[10]In contrast to the original long-run Diewert GL, where zero substitution between K and the other factors is obtained simply (and without any complications) by setting $\beta_{KL} = \beta_{KE} = 0$, see equation (12).

5. Using the long-run Diewert GL cost function generally

In this section, the results concerning the (Diewert) long-run generalized Leontief cost function of section 2B are generalized to the n factor case with k quasi-fixed factors and $l = n - k$ flexible factors. However, this section could also be regarded as a specific application of the general shadow price framework presented in section 3.

In this section, the GL cost function is extended slightly compared to section 2B, as there is no longer assumed constant returns to scale, but instead − as in Diewert's original paper − that the underlying production function is homothetic, so that Y in (11) is replaced by h(Y), where it is assumed that h(0) = 0, h′(Y) > 0 and h(Y) tends to infinity as Y tends to infinity. The homothetic GL cost function is given as (see Diewert (1971)):

$$C^* = \mathrm{h}(Y)\ \boldsymbol{P}^{0.5\prime}\ \boldsymbol{B}\ \boldsymbol{P}^{0.5}\ , \qquad \mathbf{(21)}$$

where $\boldsymbol{P}$ is a n×1 column vector of the n factor prices, $\boldsymbol{B} = [\beta_{ij}]$ is a n×n symmetric matrix of parameters, and where the square root symbol means that the square root of each element is taken. Shephard's Lemma, $\boldsymbol{X}^* = \partial C^*/\partial \boldsymbol{P}$, yields the long-run factor demands:

$$\boldsymbol{X}^* = \mathrm{h}(Y)\ \hat{\boldsymbol{P}}^{-0.5}\ \boldsymbol{B}\ \boldsymbol{P}^{0.5}\ , \qquad \mathbf{(22)}$$

where $\boldsymbol{X}$ is a n×1 column vector of factor levels, and where $\hat{\boldsymbol{P}}$ denotes the diagonalization of $\boldsymbol{P}$ into a n×n diagonal matrix. The n production factors are now divided into k quasi-fixed factors and $l = n - k$ flexible factors, so that (22) is partitioned into:

$$\begin{bmatrix} \boldsymbol{X}_k^* \\ \boldsymbol{X}_l^* \end{bmatrix} = \mathrm{h}(Y) \begin{bmatrix} \hat{\boldsymbol{P}}_k^{-0.5} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{P}}_l^{-0.5} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{kk} & \boldsymbol{B}_{kl} \\ \boldsymbol{B}_{kl}^{\prime} & \boldsymbol{B}_{ll} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_k^{0.5} \\ \boldsymbol{P}_l^{0.5} \end{bmatrix} , \quad \boldsymbol{B}_{kk} = \boldsymbol{B}_{kk}^{\prime},\ \boldsymbol{B}_{ll} = \boldsymbol{B}_{ll}^{\prime}\ . \qquad \mathbf{(23)}$$

That is,

$$\boldsymbol{X}_k^* = \mathrm{h}(Y)\ \hat{\boldsymbol{P}}_k^{-0.5}\ (\boldsymbol{B}_{kk}\boldsymbol{P}_k^{0.5} + \boldsymbol{B}_{kl}\boldsymbol{P}_l^{0.5})\ , \qquad \mathbf{(24)}$$

$$\boldsymbol{X}_l^* = \mathrm{h}(Y)\ \hat{\boldsymbol{P}}_l^{-0.5}\ (\boldsymbol{B}_{kl}^{\prime}\boldsymbol{P}_k^{0.5} + \boldsymbol{B}_{ll}\boldsymbol{P}_l^{0.5})\ . \qquad \mathbf{(25)}$$

The next step is to isolate $\boldsymbol{P_k}$ from (24) to obtain the shadow prices of the fixed factors:

$$\tilde{\boldsymbol{P}}_k = \left[(\hat{\boldsymbol{X}}_k/\mathrm{h}(Y) - \boldsymbol{B}_{kk})^{-1}\boldsymbol{B}_{kl}\boldsymbol{P}_l^{0.5} \right]^2 , \tag{26}$$

provided that the matrix $(\hat{\boldsymbol{X}}_k/\mathrm{h}(Y) - \boldsymbol{B}_{kk})$ is non-singular. The square symbol means squaring each element in the vector, and the derivation exploits that $\hat{\boldsymbol{P}}_k^{0.5}\boldsymbol{X}_k = \hat{\boldsymbol{X}}_k\boldsymbol{P}_k^{0.5}$. The matrix $(\hat{\boldsymbol{X}}_k/\mathrm{h}(Y) - \boldsymbol{B}_{kk})$ could be called the "characteristic" matrix, as its inverted counterpart, $(\hat{\boldsymbol{X}}_k/\mathrm{h}(Y) - \boldsymbol{B}_{kk})^{-1}$, describes how the levels of the quasi-fixed factors affect the flexible factors in the short run (see section 2B for a simple example). Inserting $\tilde{\boldsymbol{P}}_K$ in the place of $\boldsymbol{P}_K$ in the long-run demand equations for the flexible factors (25) yields the short-run demands for the same:

$$\boldsymbol{X}_l = \mathrm{h}(Y)\ \hat{\boldsymbol{P}}_l^{-0.5} \left[\boldsymbol{B}_{kl}^{\prime}\ (\hat{\boldsymbol{X}}_k/\mathrm{h}(Y) - \boldsymbol{B}_{kk})^{-1}\boldsymbol{B}_{kl} + \boldsymbol{B}_{ll}\right]\boldsymbol{P}_l^{0.5} . \tag{27}$$

The short-run costs are most easily found by using $C = \boldsymbol{P}_k'\boldsymbol{X}_k + \boldsymbol{P}_l'\boldsymbol{X}_l$, yielding:

$$C = \boldsymbol{P}_k^{\prime}\boldsymbol{X}_k + \mathrm{h}(Y)\,\boldsymbol{P}_l^{0.5\,\prime}\boldsymbol{B}_{ll}\boldsymbol{P}_l^{0.5} + \mathrm{h}(Y)\ \boldsymbol{P}_l^{0.5\,\prime}\boldsymbol{B}_{kl}^{\prime}\ (\hat{\boldsymbol{X}}_k/\mathrm{h}(Y) - \boldsymbol{B}_{kk})^{-1}\boldsymbol{B}_{kl}\boldsymbol{P}_l^{0.5}. \tag{28}$$

Long-run marginal costs are found by differentiating (21) with respect to Y:

$$MC^* = \mathrm{h}^{\prime}(Y)\ \boldsymbol{P}^{0.5\prime}\ \boldsymbol{B}\ \boldsymbol{P}^{0.5} . \tag{29}$$

Partitioning again into the quasi-fixed and flexible factors yields:

$$MC^* = \mathrm{h}^{\prime}(Y) \begin{bmatrix}\boldsymbol{P}_k^{0.5} & \boldsymbol{P}_l^{0.5}\end{bmatrix} \begin{bmatrix}\boldsymbol{B}_{kk} & \boldsymbol{B}_{kl} \\ \boldsymbol{B}_{kl}^{\prime} & \boldsymbol{B}_{ll}\end{bmatrix} \begin{bmatrix}\boldsymbol{P}_k^{0.5} \\ \boldsymbol{P}_l^{0.5}\end{bmatrix} . \tag{30}$$

Inserting the k shadow prices into MC^* gives the short-run marginal costs, MC, as

$$MC = \mathrm{h}^{\prime}(Y) \left[\tilde{\boldsymbol{P}}_k^{0.5\prime}\boldsymbol{B}_{kk}\tilde{\boldsymbol{P}}_k^{0.5} + 2\,\tilde{\boldsymbol{P}}_k^{0.5\prime}\boldsymbol{B}_{kl}\boldsymbol{P}_l^{0.5} + \boldsymbol{P}_l^{0.5\prime}\boldsymbol{B}_{ll}\boldsymbol{P}_l^{0.5} \right] . \tag{31}$$

If the shadow prices, $\tilde{\boldsymbol{P}}_K$ (26), are substituted into (31), MC is given as a function of Y, $\boldsymbol{P}_l$ and $\boldsymbol{X}_k$.

Finally − addressing the attention to the underlying production function − as it has been shown on the previous pages, it is not at all necessary to know the functional form of the underlying production function, but as it is quite simple to deduct an expression yielding it, it is derived below. The procedure is to assume that $n-1$ out of the n factors are fixed and subsequently derive the short-run demand for the n'th factor. As shown previously, the short-run factor demands are:

$$\boldsymbol{X_l} = \mathrm{h}(Y)\ \hat{\boldsymbol{P}}_l^{-0.5} \left[\boldsymbol{B}_{kl}^{\prime} \ (\hat{\boldsymbol{X}}_k / \mathrm{h}(Y) - \boldsymbol{B}_{kk})^{-1} \boldsymbol{B}_{kl} + \boldsymbol{B}_{ll} \right] \boldsymbol{P}_l^{0.5} . \qquad \textbf{(32)}$$

Since there is only one flexible factor, $\hat{\boldsymbol{P}}_l^{-0.5}$ and $\boldsymbol{P}_l^{0.5}$ are both scalars and cancel out yielding:

$$\boldsymbol{X_l} = \mathrm{h}(Y) \left[\boldsymbol{B}_{kl}^{\prime} \ (\hat{\boldsymbol{X}}_k / \mathrm{h}(Y) - \boldsymbol{B}_{kk})^{-1} \boldsymbol{B}_{kl} + \boldsymbol{B}_{ll} \right] . \qquad \textbf{(33)}$$

Here, $\boldsymbol{X_l}$ and $\boldsymbol{B_{ll}}$ are scalars, and $\boldsymbol{B_{kl}}$ is a $(n-1)\times 1$ column vector. This equation gives a relationship between the n production factors, X_1-X_n, and the production level, Y − that is, the underlying production function. It can be shown that (33) is equivalent to:

$$\begin{vmatrix} \hat{\boldsymbol{X}}_k / \mathrm{h}(Y) - \boldsymbol{B}_{kk} & -\boldsymbol{B}_{kl} \\ -\boldsymbol{B}_{kl}^{\prime} & \boldsymbol{X}_l / \mathrm{h}(Y) - \boldsymbol{B}_{ll} \end{vmatrix} = 0 , \qquad \textbf{(34)}$$

or

$$\left| \hat{\boldsymbol{X}} / \mathrm{h}(Y) - \boldsymbol{B} \right| = 0 . \qquad \textbf{(35)}$$

So the underlying dual "GL-production function" is given by the condition that the "full" characteristic matrix is singular; that is, that this matrix has zero determinant. Generally, Y would be given as the solution to a polynomial of degree n, so it should be stressed that (35) only gives *necessary* conditions for the production function.

6. Introducing technical progress, scale effects etc.

Up to now, I have abstracted almost completely from technical progress (and other exogenous variables) and scale effects. This has been fully deliberate, as such generalizations are very simple to implement, provided that one is willing to accept the plausibly minor assumption that the effects from technical progress (t) and the production level (Y) are purely *factor-augmenting*; that is, affecting each factor through a factor-specific efficiency-index, $e_i = \mathrm{e_i}(t, Y)$, defined below.

Starting out with a usual production function without efficiency indexes, $Y = \mathrm{F}(X_1, \ldots , X_n)$, this yields the long- run factor demand functions: $X_i^* = \mathrm{X}_i^*(Y, P_1, \ldots , P_n)$, $i = 1, \ldots , n$, and the long-run cost function $C^* = \mathrm{C}^*(Y, P_1, \ldots , P_n)$. Using the shadow price result, short-run factor demands can be directly derived from long-run factor demands (see section 3 above). If it is assumed that only X_1 is quasi-fixed, the short-run factor demands will be of the form $X_i = \mathrm{X}_i(Y, X_1, P_2, \ldots , P_n)$, $i = 2, \ldots , n$. Assigning now an efficiency index, e_i, to each factor gives a production function with disembodied *factor-augmenting* efficiency indexes, $Y = \mathrm{F}(e_1X_1, \ldots , e_nX_n)$. Here, the functional form, $\mathrm{F}(\cdot)$, is the same, and by rewriting the costs as $C = P_1X_1 + \ldots + P_nX_n = (P_1/e_1)\cdot(e_1X_1) + \ldots + (P_n/e_n)\cdot(e_nX_n)$, it is easy to prove that the following long-run factor demands result:

$$X_i^* = \frac{1}{e_i}\,\mathrm{X_i}^*\left(Y,\ \frac{P_1}{e_1},\ \ldots\ ,\ \frac{P_n}{e_n}\right), \quad i = 1, \ldots , n \ , \qquad \textbf{(36)}$$

And the following long-run cost function:

$$C^* = \mathrm{C}^*\left(Y,\ \frac{P_1}{e_1},\ \ldots\ ,\ \frac{P_n}{e_n}\right) . \qquad \textbf{(37)}$$

From (36) it is seen that the *efficiency-corrected* factor levels, $e_iX_i^*$ respond to the *efficiency-corrected* factor prices, $P_i\,/e_i$. The mathematical functions $\mathrm{X}_i^*(\cdot)$ and $\mathrm{C}^*(\cdot)$ are the same as without efficiency indexes, so the point is that it is very easy to introduce disembodied factor-augmenting technological progress (or effects from other exogenous factors) and/or scale effects into any system of long-run factor demand functions (or into any long-run cost function). Similarly, short-run factor demands are given as (assuming here, that there is only one quasi-fixed factor, X_1):

$$X_i = \frac{1}{e_i}\,\mathrm{X_i}\left(Y,\ e_1X_1,\ \frac{P_2}{e_2}\ \ldots\ ,\ \frac{P_n}{e_n}\right), \quad i = 2, \ldots , n \ , \qquad \textbf{(38)}$$

Generally, one can deduce all concepts from a "stripped down" or "raw" constant returns to scale cost or production function without technical progress/other exogenous variables, and *then* easily introduce exogenous factors and scale effects via these efficiency indexes. This is done simply by multiplying all factor levels and dividing all factor prices with the corresponding efficiency indexes, just as if the factors and the prices were pre-corrected for the efficiency. The only exception to this rule is the *marginal costs*, where it must be remembered that the efficiency indexes themselves can be functions of Y. Thus, if the efficiency indexes are Y-dependent, the expression yielding long-run marginal costs, $MC^* \equiv \partial C^*/\partial Y$, must be re-calculated in the light of this. This being done, short-run marginal costs can still be found by inserting the shadow price(s) of the quasi-fixed factor(s) into the long-run marginal cost function.

The indexes could be formulated as e.g.

$$e_i = \exp[\,\omega_{i1} t + \tfrac{1}{2}\omega_2 t^2 + \psi_{i1}\log(Y) + \tfrac{1}{2}\psi_2 \log^2(Y) + \phi\, t \log(Y)\,]\ ,\quad i = 1,... \tag{39}$$

Here, the second order effects and the cross effect between t and $\log(Y)$ are assumed identical in each of the n efficiency indexes. Most interestingly, it can be shown that if one takes a "raw" no technical progress/constant returns to scale long-run translog cost function (see Christensen/Jorgenson/Lau (1971 and 1973) or Diewert/Wales (1987) p. 46) and introduces efficiency indexes of the form (39) by means of (37) – the result is the usual translog cost function *with* technical progress and scale effects (see appendix B for proofs, more details, and a numerical example). Thus, the $2n$+3 translog-parameters that add flexibility to the "raw" constant returns to scale translog (the latter containing $n(n+1)/2$ parameters) can be exactly (mathematically) translated into the $2n$+3 parameters of (39). Since the translog cost function is fully flexible, and since (39) and the trend- and scale effects of the translog cost function turn out to be exactly the same thing, it follows that the introduction of efficiency indexes formulated as (39) adds full flexibility to any "raw" no technical progress/constant returns to scale cost function.

In (39), the expression $\omega_{i1} + \omega_2 t + \phi\log(Y)$ indicates – multiplied by 100 – by how many per cent the efficiency of factor i increases from period t to period t+1, and the expression $\psi_{i1} + \psi_2\log(Y) + \phi t$ denotes by how many percent the efficiency index increases, when the production level is increased by one percent. Other formulations might prove equally useful, and other exogenous factors in addition to time, t – such as i.e. infrastructure, education level/human capital, changes in the capital stock (representing internal costs of installing/removing capital equipment), mean age of the capital stock (capturing vintage-effects), climate, fuel-efficiency etc. – may enter the efficiency indexes as well.

Despite (39) being fully flexible, one could relax the restriction that the second order effects must be identical for the different indexes. Thus, a more general formulation would be the following:

$$e_i = \exp[\,\omega_{i1}t + \tfrac{1}{2}\omega_{i2}t^2 + \psi_{i1}\log(Y) + \tfrac{1}{2}\psi_{i2}\log^2(Y) + \phi_i\, t\log(Y)\,] \ , \quad i = 1,. \tag{40}$$

If $\omega_{i1} = \omega_1$, $\omega_{i2} = \omega_2$, and $\phi_i = \phi$, technical progress is Hicks-neutral (unbiased), and similarly, if $\psi_{i1} = \psi_1$, $\psi_{i2} = \psi_2$, and $\phi_i = \phi$, the production function is homothetic (unbiased scale effects). Specifically, if $\psi_{i1} = \psi_1$, $\psi_{i2} = 0$, and $\phi_i = 0$, the production function is homogenous of degree $1/(1–\psi_1)$. The restriction $\psi_{i1} = \psi_{i2} = \phi_i = 0$ implies constant returns to scale.

The advantage of the efficiency approach lies in two points. Firstly, it is easy to introduce these indexes into any "raw" no technical progress/constant returns to scale cost function or factor demand system without further ceremony, and secondly, and most importantly, the interpretation of the parameters is much more straightforward than e.g. trying to figure out the interpretation of a trend in a cost share or factor intensity, as one has to do with the usual translog and generalized Leontief trend formulations.

Regarding the efficiency index approach, it is also worth mentioning that the way these efficiency indexes influence the long-run demands can be decomposed using the following simple relationship:

$$\partial\log(\boldsymbol{X}^*) = -(\boldsymbol{I}+E)\,\partial\log(\boldsymbol{e}) \ , \tag{41}$$

where $\boldsymbol{X}^*$ is a $n\times1$ vector of the long-run factor levels, $\boldsymbol{I}$ is a $n\times n$ identity matrix, $\boldsymbol{E}$ is a $n\times n$ matrix of long-run partial price elasticities and $\boldsymbol{e}$ is a $n\times1$ vector of efficiency indexes. From this relationship it is seen that if there is no substitution ($\boldsymbol{E} = \mathbf{0}$), an increase in the efficiency of factor *i* by 1% simply causes a correspon-ding decrease in the use of factor *i* itself by 1%. If there is non-zero factor substitution, the use of factor *i* would fall by *less* than 1%, and this is "used" to reduce the use of one or more of the other factors as well.[11] If the formulation (40) is used, the trend- and scale effects can be decomposed into $\partial\log(\boldsymbol{X}^*)/\partial t = –(\boldsymbol{I}+\boldsymbol{E})[\Omega_1 + \Omega_2 t + \Phi\log(Y)]$, and $\partial\log(\boldsymbol{X}^*)/\partial\log(Y) = –(\boldsymbol{I}+\boldsymbol{E})[\Psi_1 + \Psi_2\log(Y) + \Phi t]$, where Ω_1, Ω_2, Ψ_1, Ψ_2 and Φ are $n\times1$ vectors of the efficiency parameters of (40).

[11]This is the normal case. However, if the substitution is very large, the use of factor *i* itself might even rise, if it gets more efficient. This would be the case if the own-price elasticity of factor *i* is below −1. Besides, a rise in the efficiency of factor *i rises* the use of those of the other factors that are complementary to *i* (negative cross-price elasticities).

7. Useful approximations between short- and long-run factor demand

As I see it, using a long-run cost function as a basis for deriving all necessary short-run concepts entails a lot of advantages, compared to using a short-run cost function. This claim is dealt with in the previous sections, but there is of course the problem of what to do, if it is not possible to find the shadow price(s) *analytically* (in closed form). This is of course unpleasant, even though one could of course always find the shadow price(s) by means of numerical methods instead. But one can not a priori be sure that the shadow price(s) can be solved analytically from an arbitrary long-run factor demand system. This is e.g. not possible regarding the factor demands derived from the long-run translog cost function or the nested CES production function, whereas it is e.g. possible in the case of the long-run generalized Leontief or long-run normalized quadratic cost functions[12]. As stated above, this problem is of course unpleasant, but it has its exact counterpart in the short-run cost function "sphere", as one can similarly not a priori be sure that the corresponding long-run concepts can be found analytically from the short-run concepts. Actually, this is not possible regarding the most common formulation of a short-run translog cost function (see e.g. Atkinson/Halvorsen (1976), or the more recent Berndt/Friedlaender/Chiang/Vellturo (1993)).

Instead of relying on numerical methods (which might complicate the estimation method considerably), one could alternatively use the following approximations between short- and long-run demands. The idea is to use a logarithmic linearization of the relationship between the factors and the factor prices around the long-run levels of the former, $\boldsymbol{X}^*$:

$$\partial\log(\boldsymbol{X}^*) = \boldsymbol{E}\,\partial\log(\boldsymbol{P}) \ . \qquad \textbf{(42)}$$

Here, $\boldsymbol{X}^*$ is a $n\times1$ vector of the n factors, $\boldsymbol{E}$ is a $n\times n$ matrix of long-run partial price elasticities and $\boldsymbol{P}$ is a $n\times1$ vector of factor prices. Partitioning (42) into the k quasi-fixed and $l = n - k$ flexible factors, the following is obtained:

$$\partial\log\begin{bmatrix}\boldsymbol{X}_k^*\\ \boldsymbol{X}_l^*\end{bmatrix} = \begin{bmatrix}\boldsymbol{E}_{kk} & \boldsymbol{E}_{kl}\\ \boldsymbol{E}_{lk} & \boldsymbol{E}_{ll}\end{bmatrix}\partial\log\begin{bmatrix}\boldsymbol{P}_k\\ \boldsymbol{P}_l\end{bmatrix} \ . \qquad \textbf{(43)}$$

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[12]Regarding the normalized quadratic cost function, see Diewert/Wales (1987) (they call it "the symmetric generalized McFadden cost function"), or see Morrison (1986). Using this functional form, the shadow prices are most easily computed if none of the quasi-fixed factors are at the same time selected to be the (asymmetric) "normalizing" factor. For the first example of using a *short-run* normalized quadratic cost function, see Berndt/Fuss/Waverman (1980).

This equation expresses how the levels of $\boldsymbol{X}_k^*$ and $\boldsymbol{X}_l^*$ change, when e.g. the prices of the quasi-fixed factors, $\boldsymbol{P}_k$, change. As shown in table 2 in section 3, the shadow price result implies changing the prices of the quasi-fixed factors, $\boldsymbol{P}_k$, until the levels of the quasi-fixed factors, $\boldsymbol{X}_k^*$, are equal to their predetermined levels, $\boldsymbol{X}_k$. From (43) it follows that these virtual/shadow prices, called $\tilde{\boldsymbol{P}}_K$, can be approximated as (please note in the two following formulas that the shadow price method leaves $\boldsymbol{P}_l$ unaltered):

$$\log(\tilde{\boldsymbol{P}}_k) - \log(\boldsymbol{P}_k) \approx \boldsymbol{E}_{kk}^{-1} \left[\log(\boldsymbol{X}_k) - \log(\boldsymbol{X}_k^*) \right], \qquad \textbf{(44)}$$

provided that the submatrix $\boldsymbol{E}_{kk}$ is non-singular. According to the shadow price result, the long-run demands for the flexible factors evaluated at the shadow prices of the quasi-fixed factors yield the short-run demands for the former. That is (cf. (43)),

$$\log(\boldsymbol{X}_l) - \log(\boldsymbol{X}_l^*) \approx \boldsymbol{E}_{lk} \left[\log(\tilde{\boldsymbol{P}}_k) - \log(\boldsymbol{P}_k) \right], \qquad \textbf{(45)}$$

Inserting (44) into (45) results in

$$\log(\boldsymbol{X}_l) \approx \log(\boldsymbol{X}_l^*) + \boldsymbol{E}_{lk} \boldsymbol{E}_{kk}^{-1} \left[\log(\boldsymbol{X}_k) - \log(\boldsymbol{X}_k^*) \right]. \qquad \textbf{(46)}$$

These are *approximated* short-run factor demands for the flexible factors ($\boldsymbol{X}_l$), but it must be made clear, that the approximation may be less good far from the long-run levels. But under normal circumstances − that is, provided that the quasi-fixed factors do not deviate too much from their long-run levels − the formula should be useful for most practical purposes.[13]

To illustrate the relationship, consider the case of only one quasi-fixed factor, K, and two flexible factors, L and E. Around the long-run levels, K^*, L^* and E^*, the

[13]Regarding short-run marginal costs the following approximation may be used, provided here, that there is constant returns to scale. With constant returns to scale, MC^* can be approximated by

$$\partial\log(MC^*) = \boldsymbol{s}_k^{*\prime} \partial\log(\boldsymbol{P}_k) + \boldsymbol{s}_l^{*\prime} \partial\log(\boldsymbol{P}_l),$$

where $\boldsymbol{s}_k^*$ and $\boldsymbol{s}_l^*$ are vectors of the long-run cost shares. This is simply a logarithmic version of Shephard's Lemma, since $MC^* = AC^*$ under constant returns to scale. The short run costs, MC, are found by evaluating the above equation at the k shadow prices. This gives the approximation

$$\log(MC) \approx \log(MC^*) + \boldsymbol{s}_k^{*\prime} \boldsymbol{E}_{kk}^{-1} \left[\log(\boldsymbol{X}_k) - \log(\boldsymbol{X}_k^*) \right].$$

formula implies that $\log(L) \approx \log(L^*) + e_{LK}/e_{KK}\,[\log(K) - \log(K^*)]$, and $\log(E) \approx \log(E^*) + e_{EK}/e_{KK}\,[\log(K) - \log(K^*)]$, where e_{ij} is the long-run partial price elasticity of factor i with respect to factor price j. If e.g. e_{EK} is negative, entailing complementarity between K and E, it is seen that more K raises the demand for E, too, as one would expect.

The short run costs, C, differentiated with respect to the quasi-fixed factors, $\boldsymbol{X_k}$, is given as (see Squires (1994), p. 238):

$$\frac{\partial C}{\partial \boldsymbol{X_k}} = \boldsymbol{P_k} - \boldsymbol{\tilde{P}_k} \ . \qquad \textbf{(47)}$$

This concept is often used in the context of *adjustment costs*; that is, adding the time-differences of the quasi-fixed factors to the cost function (usually in the form of "internal" or "external" costs, see e.g. Denny/Fuss/Wavermann (1981) or Morrison (1988)). This approach aims at explaining the rigidities of the quasi-fixed factors, and the results often make use of the second derivatives of the short-run costs with respect to the quasi-fixed factors, as these expressions can be used to form approximative adjustment paths. Around the long run levels of the quasi-fixed factors, the second derivatives can be approximated by the following formula (simply rewriting (44)):

$$\frac{\partial^2 C}{\partial \boldsymbol{X_k}^2} = -\frac{\partial \boldsymbol{\tilde{P}_k}}{\partial \boldsymbol{X_k}} \approx -\boldsymbol{\hat{P}_k}\,\boldsymbol{E_{kk}^{-1}}\,(\boldsymbol{\hat{X}_k^*})^{-1} \ , \qquad \textbf{(48)}$$

where the hat denotes diagonalization of the vector. Again, if there were only one quasi fixed factor, K, the formula would say that $\partial^2C/\partial K^2 \approx -P_K/(e_{KK}\,K^*)$ around $C = C^*$, so that if the own-price elasticity of K, e_{KK}, is very low (very low substitutability), the short run cost function bends very strongly around $C = C^*$. This implies that small deviations between K and K^* makes C exceed C^* by a large amount.

To complete the picture, it should be noted that this paper does not deal with the problem of explaining *why* the quasi-fixed production factors are not flexible – and *how* the quasi-fixed factors adjust over time. This would imply discussing among other things adjustment costs and uncertainty, and going into that discussion would carry us too far. I would instead refer the interested reader to Nickell (1985) or the outstanding Galeotti (1996), and only note that Nickell's paper shows that under some reasonable assumptions regarding the short-run cost function, adjustment costs, and expectation rules, dynamic optimization implies that the quasi-fixed factors adjust to their long-run levels by means of simple error correction mechanisms.

8. An example using the Berndt-Wood data

To illustrate the above techniques, some estimations on the much used Berndt-Wood data set (covering US manufacturing over the period 1947-71) are presented. In the estimations capital (*K*), labour (*L*), energy (*E*) and materials (*M*) are described as a function of the four corresponding factor prices, the production level (*Y*), and time (*t*).[14]

Diewert/Wales (1987) use the same data set to estimate – among other things – translog- and generalized Leontief cost functions, with full flexibility regarding price elasticities, trend- and scale effects. As I see it, however, these estimations yield implausible scale effects, especially for *K* and *E,* probably because log(*Y*) and *t* are correlated with a correlation coefficient of 0.971, so that trend- and scale parameters become difficult to identify separately.

In order to avoid these unpleasant multicollinearity problems and obtain more easily interpretable results, I use a *constant returns to scale* long-run (Diewert) generalized Leontief cost function. Regarding technical progress, the efficiency indexes of section 6 are implemented, without *Y*-effects, and using unrestricted t^2-terms; that is, two trend parameters in each of the four efficiency indexes. The following estimation is static, assuming $K = K^*$, $L = L^*$, $E = E^*$ and $M = M^*$.[15]

Table 3. Estimation of a static factor demand system derived from a long-run generalized Leontief cost function

	Long-run partial price elasticities				Growth of eff. ind.		Adjustment				
	P_K	P_L	P_E	P_M	1949	1971	λ_1	λ_2	SEE	DW	JB
K	−0.18	0.29	−0.09	–0.02	0.6%	1.5%	1.00	1.00	6.5%	0.83	0.8
L	0.04	−0.12	0.09	–0.01	0.1%	1.4%	1.00	1.00	2.6%	1.16	0.1
E	−0.09	0.59	−0.50	0.00	0.8%	3.5%	•	•	5.1%	0.91	1.8
M	0.00	0.00	0.00	0.00	0.4%	0.2%	•	•	1.4%	1.45	6.6

Note: The estimation period is 1948-71, and the price elasticities are of the type $\partial\log(X_i)/\partial\log(P_j)$. JB is the Jarque/Bera test for normality of the residuals and should not exceed $\chi^2_{95\%}(2) = 6.0$. Log likelihood = 228.83. The cost function is not concave at any data point.

[14]See Berndt/Wood (1975) and Berndt/Khaled (1979) regarding the construction of the data.

[15]More precisely, equation (22) with h(*Y*) = *Y* is used. All of these four demand equations are extended with efficiency indexes in the manner shown in (36), and logarithms are taken on both sides of the equality signs. The efficiency indexes are of the form (40) with the last three terms suppressed. The estimation procedure is maximum likelihood, assuming that the disturbance terms follow a multivariate normal distribution with zero means and constant variances and covariances.

The 4×4 numbers to the left are long-run partial price elasticities (evaluated in 1971), where it is among other things seen that *K* and *E* are complementary. The two columns of growth rates of the efficiency indexes (in 1949 and 1971, respectively) show e.g. that the efficiency growth of *K* changed from 0.6% p.a. in 1949 to 1.5% p.a. in 1971. The Jarque/Bera tests do not indicate serious problems with the assumption of the error terms being normally distributed, whereas the Durbin-Watson tests are not very convincing. To a large extent, the elasticities resemble the elasticities found in Diewert/Wales (1987) (table 4, rows 3 and 4), where there are similar problems of *M* being fully or almost price insensitive (corresponding to *M* being strongly separable).

Dynamizing this system – here abstracting from possible cross-effects in the adjustment of *K* and *L* – it is now assumed that *K* and *L* adjust to their long-run levels according to the following simple error correction mechanism (here for *K*):

$$\Delta \log(K_t) \ = \ \lambda_1 \, \Delta \log(K_t^*) \ + \ \lambda_2 \left[\log(K_{t-1}^*) - \log(K_{t-1}) \right] , \qquad \textbf{(49)}$$

where λ_1 and λ_2 are adjustment parameters (see the end of the preceding section for a justification of using an error correction mechanism). Utilizing the shadow price results of section 5 to obtain short-run factor demand equations for *E* and *M*, the following result is obtained:[16]

Table 4. Estimation of a factor demand system derived from a long-run generalized Leontief cost function, with *K* and *L* quasi-fixed and *E* and *M* flexible

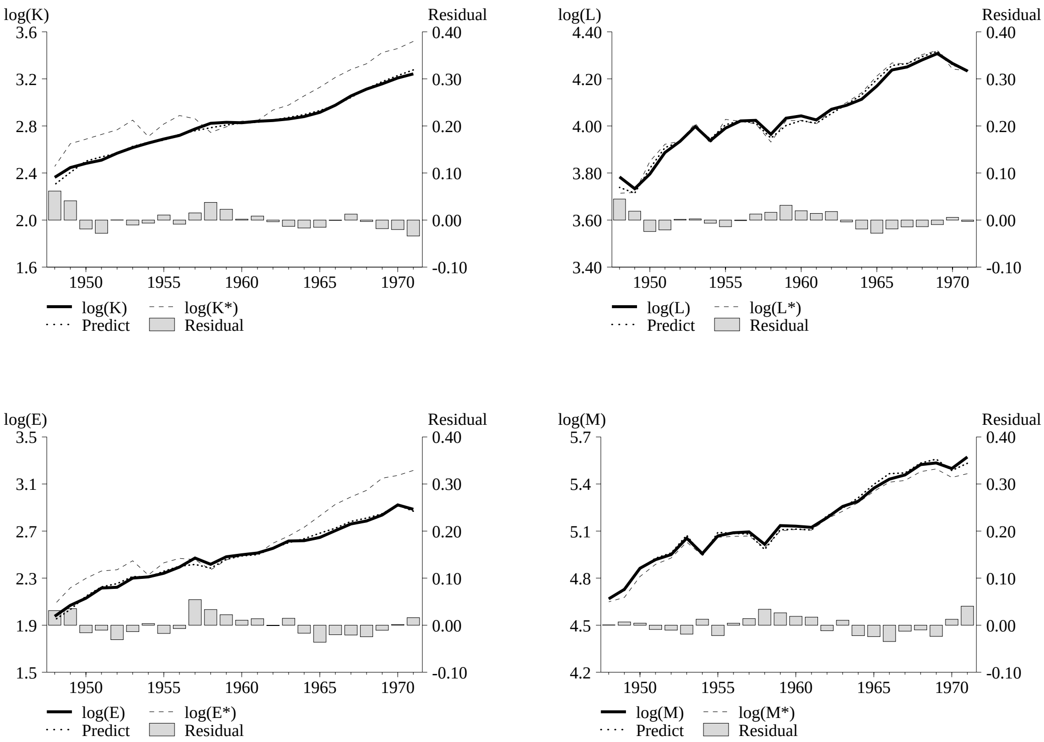
	Long-run partial price elasticities				Growth of eff. ind.		Adjustment				
	P_K	P_L	P_E	P_M	1949	1971	λ_1	λ_2	SEE	DW	JB
K	−0.49	0.50	−0.34	0.33	12.4%	−13.9%	0.10	0.25	2.2%	0.73	4.1
L	0.11	−0.31	0.04	0.17	–0.6%	3.5%	0.78	1.03	1.8%	0.65	1.2
E	−0.34	0.16	−0.41	0.59	12.6%	–15.8%	•	•	2.3%	0.91	1.4
M	0.04	0.09	0.07	−0.19	–1.2%	2.6%	•	•	2.0%	1.00	0.8

Note: The estimation period is 1948-71, and the price elasticities are of the type $\partial\log(X_i)/\partial\log(P_j)$. JB is the Jarque/Bera test for normality of the residuals and should not exceed $\chi^2_{5\%}(2) = 6.0$. Log likelihood = 265.66. The cost function is concave at all data points.

[16]More precisely, equation (22) with h(*Y*) = *Y* is used as regards K^* and L^*, and equation (27) with h(*Y*) = *Y* is used as regards *E* and *M*. All of these four demand equations are extended with efficiency indexes in the manner shown in (36) and (38), respectively, and logarithms are taken on both sides equality signs in the *E*- and *M*-equations. The efficiency indexed are of the form (40) with the last three terms suppressed.

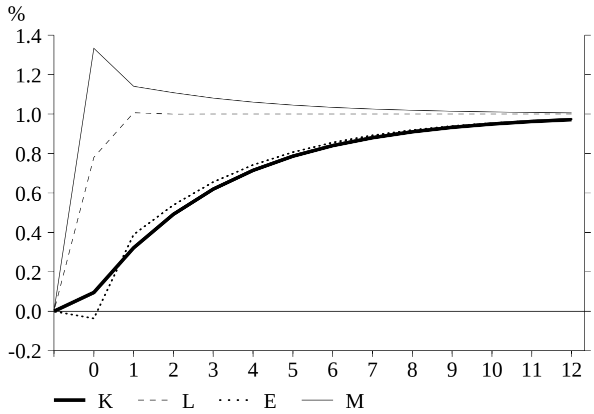
From table 4 it is seen that the DW-tests are still not convincing, but it must be kept in mind that the dynamic formulation is very simple with only four adjustment parameters. The log likelihood value increases significantly from 228.83 to 265.66 (comparing twice the difference = 73.66 with $\chi^2_{95\%}(4) = 9.49$), most notably due to the improved SEE of the residuals of *K* and *E*. Furthermore, *M*'s own-price elasticity grows (numerically) to –0.19, and the cost function is now monotonous and concave in the prices at all data points. In the dynamic estimation, the growth rates of the *K*- and *E*-efficiencies have been decreasing over the estimation period, hinting that technological progress and capital/energy could be conceived of as being complementary in the second half of the estimation period. In the following figure, the historical fit is depicted:

Figure 3. Historical fit, dynamic model



In the figure, it can among other things be seen that K^* exceeds *K* in most of the estimation period (the average gap being 17%), which is reflected in E^* also exceeding *E* (due to the complementarity between *K* and *E*). If the production level, *Y*, is increased by 1% in the dynamic model, the adjustment looks as follows:

Figure 4. The dynamic effect of a 1% increase in the production level



In the long run, all factors are also increased by 1%, due to the imposed constant returns to scale. In the short run, however, *K* and *L* react sluggishly, and this causes *M* to overshoot in the short run, whereas *E* follows *K* quite closely due to the complementarity of the two factors. It can be tested whether *L* is also flexible, but this is not so, confirming the hypothesis of firms performing "labour-hoarding" in the short run. The short-run (first year) effect on *K* and *L* is 0.10% and 0.78%, respectively, indicating that labour is moving much more rapidly than capital, whereas the short-run effects on *E* and *M* are –0.04% and 1.33% respectively.[17]

All in all the dynamic estimation compares favourably with the static estimation, apart maybe from the efficiency indexes of *K* and *E* changing a bit rapidly. The effects of the efficiencies on factor demand is, however, propagated through the matrix of price elasticities (cf. equation (41) in section 6), and the effects on factor demand and costs are much less different, as shown below:

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[17]To illustrate one of the approximation formulas of section 7, the last-mentioned effects on the flexible factors (*E* and *M*) can be calculated as follows, using only the effects on *K* and *L* together with the relevant submatrices of the matrix of long-run partial price elasticities (cf. equation (46)):

$$\begin{pmatrix}-0.04\\ 1.33\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}-0.34 & 0.16\\ 0.04 & 0.09\end{pmatrix}\begin{pmatrix}-0.49 & 0.50\\ 0.11 & -0.31\end{pmatrix}^{-1}\cdot\left[\begin{pmatrix}0.10\\0.78\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}\right],$$

provided that the incoming numbers are not rounded off as it is done in table 3. Hence, the long-run partial price elasticities fully determine how a disequilibrium in the fixed factors translates into a disequilibrium in the flexible factors.

Table 5. Effects of technological change on inputs and total costs

	Static		Dynamic	
	1948	1971	1948	1971
K	–0.6	–1.3	–2.0	–0.8
L	–0.9	–1.6	–1.1	–0.8
E	–0.9	–2.5	–1.7	2.6
M	–0.4	–0.2	–0.1	–0.8
C	–0.6	–0.8	–0.6	–0.6

Note: Table entries are $100 \cdot \partial \log(X_i)/\partial t$ for input i and $100 \cdot \partial \log(C)/\partial t$ for total costs.

From the table, it can among other things be seen that the annual decrease of total costs due to technological progress has been between 0.6% to 0.8% in both models.

9. Conclusion

In this paper, it has been shown that it *is* possible to start out with a long-run cost function and derive all necessary short-run concepts by the means of shadow or virtual prices. This approach entails many theoretical and practical advantages, apart from clarifying the relationship between the short and the long run considerably. And if the shadow prices are not analytically computable, some very easy and useful approximations are provided, making use of long-run partial price elasticities.

As an example, using the original long-run Diewert generalized Leontief cost function, one can analytically compute all the usually employed short-run concepts, such as short-run factor demands, costs and marginal costs by means of shadow prices (see sections 2B and 5). This makes the generalized Leontief cost function a quite promising candidate for applied econometric work, if one wants to make sure that the short-run behavior (short-term dynamic adjustment) is in agreement with standard neoclassical theory.

However, the shadow price approach presented here might be useful in many other cases, as it is by no means restricted to work only with e.g. the generalized Leontief cost function. The shadow price approach may be used on *any* system of long-run factor demand functions, regardless of whether these are constructed by means of production or cost functions, and regardless of whether the cost function is e.g. a translog or a generalized Leontief or something completely different.

As to technical progress or other exogenous factors, scale effects, etc., these can be easily introduced into any cost function or factor demand system by means of factor augmenting efficiency indexes. It is shown that the trend- and scale parameters of the translog cost function can be exactly translated into corresponding parameters in the above-mentioned efficiency indexes. All in all, the efficiency methodology seems very promising, as it is both fully flexible and very easy to understand and interpret.

Finally, it is shown in section 8 that it is possible to estimate a quite plausible factor demand system (with four production factors, of which two are assumed quasi-fixed) on the Berndt-Wood data set, using efficiency indexes together with the "original" (Diewert) long-run generalized Leontief, and using shadow prices to deduct short-run factor demands for the two flexible factors. The results indicate that much profit (both regarding fit and economic properties) can be obtained from using a dynamic factor demand system compared to a static one.

All in all, I feel that the results presented in this paper could prove extremely useful in applied econometric work, because the shadow price approach provides an easy and very transparent way of relating short- and long-run concepts in a consistent manner, in contrast to e.g. basing the work on short-run cost functions, where the derivations are often very complex and confusing, and where one is all but sure about the global properties of the involved functional forms. In addition to that, using efficiency indexes also seems to be a very convenient and easily interpretable way of introducing technical progress/other exogenous factors, scale effects etc., into any factor demand system.

Appendix A: Proof of shadow price result regarding marginal costs

This appendix contains a proof that short-run marginal costs are found by inserting the shadow price into the relation for the long-run marginal costs.

The short-run costs observe the following relationship (see Squires (1994), p. 238):

$$C = \tilde{C}^{*} + \sum_{i=1}^{k} (P_i - \tilde{P}_i) X_i \ , \tag{A1}$$

where X_i is factor i, P_i its factor price. Defining $\tilde{C}^*$ as C^* evaluated at the k shadow prices,

$$\tilde{C}^{*} = \mathrm{C}^{*}(Y,\ \tilde{P}_1,\ \dots\ ,\ \tilde{P}_k,\ P_{k+1},\ \dots\ ,\ P_n) \ . \tag{A2}$$

Since $\mathrm{d}X_i/\mathrm{d}Y = 0$, $i = 1, \dots , k$, it follows that

$$\frac{\mathrm{d}C}{\mathrm{d}Y} = \frac{\mathrm{d}\tilde{C}^{*}}{\mathrm{d}Y} - \sum_{i=1}^{k} \frac{\partial \tilde{P}_i}{\partial Y} X_i$$

$$= \left(\frac{\partial \tilde{C}^{*}}{\partial Y} + \sum_{i=1}^{k} \frac{\partial \tilde{C}^{*}}{\partial \tilde{P}_i} \frac{\partial \tilde{P}_i}{\partial Y} \right) - \sum_{i=1}^{k} \frac{\partial \tilde{P}_i}{\partial Y} X_i = \frac{\partial \tilde{C}^{*}}{\partial Y} \ , \tag{A3}$$

as $\partial\tilde{C}^*/\partial\tilde{P}_i = X_i$ using Shephard's Lemma. Since the long-run marginal costs, MC^*, are given as

$$MC^{*} \equiv \frac{\partial C^{*}}{\partial Y} = \mathrm{MC}^{*}(Y,\ P_1,\ \dots\ ,\ P_n) \ , \tag{A4}$$

the short-run marginal costs, MC, are given by:

$$MC \equiv \frac{\mathrm{d}C}{\mathrm{d}Y} = \frac{\partial \tilde{C}^{*}}{\partial Y} = \mathrm{MC}^{*}(Y,\ \tilde{P}_1,\ \dots\ ,\ \tilde{P}_k,\ P_{k+1},\ \dots\ , P_n) \ . \tag{A5}$$

That is, the long-run marginal costs, evaluated at the k shadow prices, are identical to the short-run marginal costs.

Appendix B: Proof regarding the translog cost function and efficiency indexes

The usual long-run translog cost function is given as (here with no technical progress/constant returns to scale, see e.g. Christensen/Jorgenson/Lau (1971 and 1973) or Diewert/Wales (1987)):

$$\log(C^*) = a_0 + \log(Y) + \log(\boldsymbol{P}')\boldsymbol{A} + \tfrac{1}{2}\log(\boldsymbol{P}')\boldsymbol{B}\log(\boldsymbol{P}), \qquad \textbf{(B1)}$$

where C^* is the long-run costs, Y the production level, and $\boldsymbol{P}$ a vector of factor prices. As to the parameters, a_0 is a constant, $\boldsymbol{A}$ is a $n\times 1$ vector of parameters summing to unity, and $\boldsymbol{B}$ is a $n\times n$ symmetric matrix of parameters with rows and columns summing to zero. It is now assumed that the efficiency indexes are formulated as follows:

$$\log(\boldsymbol{e}) = \boldsymbol{\Omega}_1 t + \tfrac{1}{2}\boldsymbol{\Omega}_2 t^2 + \boldsymbol{\Psi}_1\log(Y) + \tfrac{1}{2}\boldsymbol{\Psi}_2\log^2(Y) + \boldsymbol{\Phi}\, t\log(Y) \qquad \textbf{(B2)}$$

where Ω_1 and Ψ_1 are $n\times 1$ vectors of parameters, and Ω_2, Ψ_2 and Φ are $n\times 1$ vectors of identical parameters (cf. (39)). Introducing the efficiency indexes (B2) into (B1) by means of (37), and utilizing that $\boldsymbol{B}\,\Omega_2 = \boldsymbol{B}\,\Psi_2 = \boldsymbol{0}$ since the rows and columns of $\boldsymbol{B}$ sum to zero, the following is obtained:

$$\begin{aligned}\log(C^*) &= a_0 + \boldsymbol{A}'\log(\boldsymbol{P}) + a_y\log(Y) + a_t t \\ &+ \tfrac{1}{2}\log(\boldsymbol{P}')\boldsymbol{B}\log(\boldsymbol{P}) + \boldsymbol{B}_y'\log(\boldsymbol{P})\log(Y) + \boldsymbol{B}_t'\log(\boldsymbol{P})\,t \\ &+ \tfrac{1}{2}a_{yy}\log^2(Y) + b_{ty}\,t\log(Y) + \tfrac{1}{2}a_{tt}t^2\end{aligned} \qquad \textbf{(B3)}$$

where

$$\begin{aligned} a_t &= -\boldsymbol{\Omega}_1'\boldsymbol{A} \\ a_y &= 1 - \boldsymbol{\Psi}_1'\boldsymbol{A} \\ \boldsymbol{B}_t &= -\boldsymbol{B}\boldsymbol{\Omega}_1 \\ \boldsymbol{B}_y &= -\boldsymbol{B}\boldsymbol{\Psi}_1 \\ a_{tt} &= \boldsymbol{\Omega}_2'\boldsymbol{A} + \boldsymbol{\Omega}_1'\boldsymbol{B}\boldsymbol{\Omega}_1 \\ b_{ty} &= -\boldsymbol{\Phi}'\boldsymbol{A} + \boldsymbol{\Psi}_1'\boldsymbol{B}\boldsymbol{\Omega}_1 \\ a_{yy} &= -\boldsymbol{\Psi}_2'\boldsymbol{A} + \boldsymbol{\Psi}_1'\boldsymbol{B}\boldsymbol{\Psi}_1 \end{aligned} \qquad \textbf{(B4)}$$

Here, a_t, a_y, a_{tt}, b_{ty} and a_{yy} are scalars, whereas $\boldsymbol{B}_t$ and $\boldsymbol{B}_y$ are $n\times 1$ vectors, where the parameters sum to zero since the rows and columns of $\boldsymbol{B}$ sum to zero. These parameters are the usual translog trend- and scale parameters. One can go the opposite way from the usual translog-parameters to the efficiency parameters in the following steps: $\boldsymbol{\Omega}_1$ is determined from the a_t- and $\boldsymbol{B}_t$-equations, $\boldsymbol{\Psi}_1$ from the a_y- and $\boldsymbol{B}_y$-equations, $\boldsymbol{\Omega}_2$ from the a_{tt}-equations, $\boldsymbol{\Psi}_2$ from the a_{yy}-equations, and $\boldsymbol{\Phi}$ from the b_{ty}-equation (when finding $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Psi}_1$, note that $\boldsymbol{B}$ can not be inverted since it does not have full rank. Note also that in the Cobb-Douglas special case of $\boldsymbol{B} = \mathbf{0}$, the equations do not have any solution, if $\boldsymbol{B}_t \neq \mathbf{0}$ or $\boldsymbol{B}_y \neq \mathbf{0}$. If $\boldsymbol{B}_t = \boldsymbol{B}_y = \mathbf{0}$, there is no unique solution, however).

Since (B3) is the normal way of introducing trend- and scale effects into a translog cost function (see e.g. Diewert/Wales (1987)), it is hereby shown that the normal translog cost function is mathematically equivalent to using a "raw" no technical progress/no scale effects translog cost function and augmenting it with the efficiency indexes of (B2). Since the usual translog cost function is flexible, it follows that efficiency indexes of the form (B2) are capable of adding flexibility to *any* "raw" no technical progress/no scale effects cost function.

To illustrate, it is shown below how to translate the estimated translog trend- and scale parameters from Diewert/Wales (1987) into the equivalent efficiency parameters. Diewert/Wales use the Berndt-Wood data set (with n = 4 production factors, K, L, E and M), and estimate the following $21 = n(n+1)/2 + 2n + 3$ translog-parameters:

Table B1. Parameter estimates, standard translog, Diewert/Wales (1987)

Parameter	Estimate	Parameter	Estimate
a_0	6.488	a_t	0.01029
a_1	0.2984	a_y	0.6204
a_2	0.04958	b_{t1}	–0.0002000
a_3	0.04611	b_{t2}	0.001228
b_{11}	0.1387	b_{t3}	0.0007784
b_{12}	0.01271	b_{y1}	–0.03102
b_{13}	0.008168	b_{y2}	–0.04075
b_{22}	0.03440	b_{y3}	–0.02874
b_{23}	–0.007818	a_{tt}	0.001179
b_{33}	0.01501	b_{ty}	–0.01294
		a_{yy}	0.01633

Note: The "missing" parameters (a_4 etc.) can be found from the the restrictions $\Sigma a_i = 1$, $\Sigma b_{ti} = 0$, $\Sigma b_{yi} = 0$, $\Sigma b_{ij} = \Sigma b_{ji} = 0$, i = 1-4.

Unfortunately, Diewert/Wales do not report their parameter estimates, so I have reproduced their estimation. It should be noted that I have scaled the variables so that the four factor prices are equal to 1 in 1971, and so that $Y = 1$ and time, $t = 0$. The latter scalings ensures that the efficiency indexes are equal to 1 in 1971. This is for convenience only. Using (B4), the $11 = 2n + 3$ trend- and scale parameters in the right part of table B1 can be translated into the following efficiency parameters:

Table B2. The parameters of table B1, translated into efficiency parameters

		t	t^2	$\log(Y)$	$\log^2(Y)$	$t\log(Y)$
K	$\log(e_1)$	0.005525	–0.001032	0.05900	0.1431	0.008255
L	$\log(e_2)$	–0.06756	–0.001032	2.093	0.1431	0.008255
E	$\log(e_3)$	–0.09701	–0.001032	3.163	0.1431	0.008255
M	$\log(e_4)$	–0.006799	–0.001032	0.1855	0.1431	0.008255

Note: The parameters in the t^2-, $\log^2(Y)$- and $t\log(Y)$-columns are identical, so that the table contains $11 = 2n + 3$ independent parameters.

Since t and Y are scaled so that they are equal to 0 and 1 in 1971, respectively, it is seen that the efficiency of the four factors had an annual growth rate due to time alone of 0.6%, –6.8%, –9.7% and –0.7% in 1971 (cf. the t-column). These annual growth rates are reduced by 0.103 %-points per year, and by 0.826 %-points per 1 percent change in Y (cf. the t^2-and $t\log(Y)$-columns). In 1971, the efficiency of the four factors grows with 0.06%, 2.09%, 3.16% and 0.19%, respectively, when Y is increased by 1% (cf. the $\log(Y)$-column). These elasticities grow by 0.14%-points, when Y grows by 1%, and by 0.00826%-points per time period (cf. the $\log^2(Y)$- and $t\log(Y)$-columns). As indicated in the text – due to t and Y being highly correlated – the scale effects in the estimation does not seem very sensible. This is seen by the fact that in 1971 the efficiencies of L and E grow by 2.09% and 3.16%, respectively, when Y grows by 1%.

In 1971, the scale elasticities of the factors themselves are –0.20, 0.52, 0.00 and 0.79, respectively, hinting that something is "wrong" with the Y-effects of K and E. However, when the parameters are translated into efficiency parameters, it is seen (table B2) that the scale-elasticity of the K-efficiency was very small (0.06) in 1971. So the strange output-elasticity of K stems in reality from the strange scale-elasticities of the L- and E-efficiencies. (See (41) and the following explanations regarding the relationship between efficiency indexes and long-run factor demand).

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